Scale, decreasing types, and extending functions continuously in o-minimal theories

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Background

- We work here with o-minimal structures (the real field is the classical example).
- A structure is o-minimal iff it is linearly ordered and any definable subset is a finite union of points and intervals.
- In an o-minimal structure, M, for any definable n-ary function, there exists a decomposition of M^n into finitely many definable "cells" such that the function is continuous on each cell.
- A consequence: every definable function in an o-minimal structure is "eventually" continuous, monotone, and unchanging in sign.
- To verify that a function is continuous on a definable set, it suffices to show that, for any two definable curves with the same endpoint, the limit of the function along the curves is the same.

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Extending Functions to Closures[1]

Let γ be a curve in M^n with one endpoint the origin, and let f be an M-definable bounded n-ary function. Can we find an initial segment of γ and a definable set containing that initial segment on which f is continuous, or extends continuously?

Under reasonable assumptions, we can find a definable set containing $\gamma \setminus \{0\}$ on which f is continuous. The difficulty is in extending f continuously to 0, which is equivalent to extending f continuously to the closure of the definable set.

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Definable γ works

Example

Let $f(x, y) = \min(1, y/x)$, and let γ be any definable curve in the first quadrant with left endpoint 0.

We can take a pair of definable curves whose derivatives at 0 are the same as the curve's at 0, giving us a cell on which f extends continuously to the closure.

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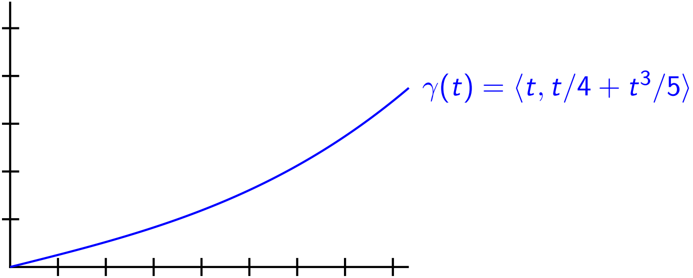

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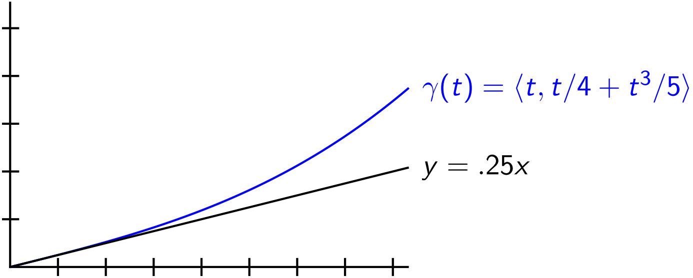

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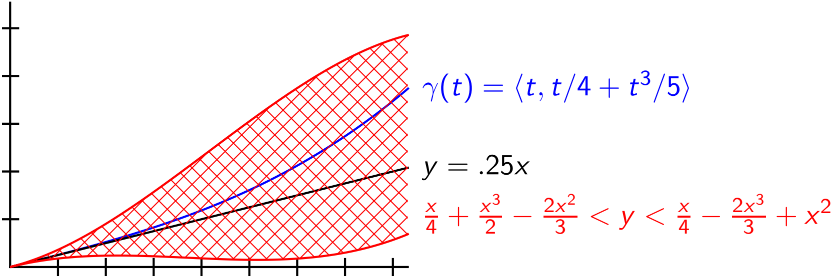

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What about the question for non-definable curves? Given a (non-definable) curve, can we find a set on which the function is continuous, which contains the curve, and on whose closure the function extends continuously.

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Good Curves

Definition

Let γ be a (not necessarily definable) curve. Say that γ is non-oscillatory if, for each definable function f from M^{m+1} to M, there exists $t_f > 0$ such that either $f(t, \gamma(t)) = 0$ for all $t \in (0, t_f)$ or $f(t, \gamma(t)) \neq 0$ for all $t \in (0, t_f)$.

Non-oscillatory Failure

- Unfortunately, requiring that γ be non-oscillatory is not enough to make any bounded definable function continuous on γ's closure.
- Let $M = (\mathbb{R}, +, \cdot, <, 0, 1)$. Let $f(x, y)$ be $\min(1, y/x)$, and let $\gamma(t) = \langle t, -t/\ln t \rangle$, so γ is undefinable in M. Note, though, that since γ is definable in the o-minimal expansion of M, $(\mathbb{R}, +, \cdot, <, \exp)$, γ is certainly non-oscillatory.
- $f(\gamma(t)) = -1/\ln t$, so $\lim_{t \to 0^+} f(\gamma(t)) = 0$.

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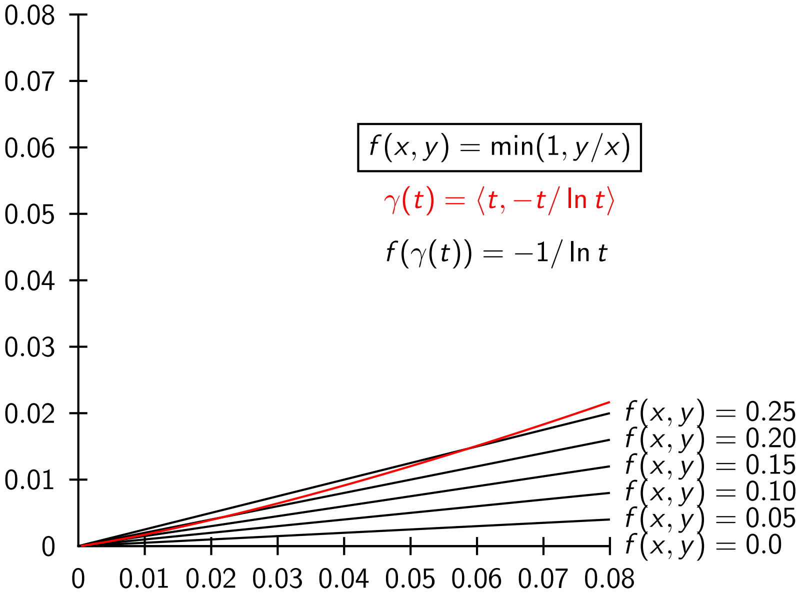
γ Is Less Than Every Linear Function



γ Cannot Be Squeezed

- $-1/\ln t$ goes to 0, but it is also greater than t^d, for any $d > 0$, for sufficiently small t. Thus, $-t/\ln t$ is greater than t^{1+d} for every $d > 0$, for sufficently small t.
- It is not hard to see that any definable set in $(\mathbb{R}, +, \cdot, <, 0, 1)$ that contains γ must contain the curve $\langle t, at \rangle$, for some real positive a.
- f cannot be continuously extended onto this set's closure, because along γ, its limit at the origin is 0, while along the linear curve, it is a.

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Why Did γ Fail?

The failure of γ can be seen as coming from the fact that we could not squeeze γ sufficiently. The gap between a linear function and a higher-power function is too great. To more closely analyze this, we can abstract out the "type" of γ.

The Limit Type of a Curve

Lemma

Let $\gamma = \langle \gamma_1, \ldots, \gamma_k \rangle$ *be a non-oscillatory curve. Let* $\gamma(t)$ *denote the sequence* $\langle \gamma_1(t), \ldots, \gamma_k(t) \rangle \in M^k$*, for* $t \in M$*. Then* $\lim_{t \to 0^+} \mathrm{tp}(\gamma(t)/M)$ *exists, in the following sense: for each formula* $\psi(x_1, \ldots, x_k)$ *in* M*, there is some* $s > 0$ *such that either* $\psi(\gamma(t))$ *holds for all* $t \in (0, s)$*, or* $\neg\psi(\gamma(t))$ *holds for all* $t \in (0, s)$*.*

Curve Limit Type Determines Definable Set Membership

Definition

With γ as above, let $\text{tp}(\gamma/M)$ denote $\lim_{t \to 0^+} \text{tp}(\gamma(t)/M)$. We can then talk about the type of γ_i over $\gamma_{<i}M$.

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Let γ be a non-oscillatory curve. Then, for any definable C, there exists an $s > 0$ such that $\gamma((0, s)) \subseteq C$ if and only if $C \in \text{tp}(\gamma/M)$.

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Back to Bad γ

We now return to $\gamma(t) = \langle t, -t/\ln t \rangle$, and examine $\mathrm{tp}(\gamma)$.

- For every $r > 0 \in \mathbb{R}_+$, $x_1 < r$ is in $\mathrm{tp}(\gamma)$.
- For every $r \in \mathbb{R}_+$, $x_2 < rx_1$ is in $\mathrm{tp}(\gamma)$.
- For every $r \in \mathbb{R}$, $q \in \mathbb{Q}_+$, $x_2 > rx_1^{1+q}$ is in $\mathrm{tp}(\gamma)$.

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Failure of Continuity Extension for a Type

Since we have equivalence of definable set membership for curves and their types, we can rephrase our failure with γ as follows:

Example

Take our model to be $(\mathbb{R}, +, \cdot, <, 0, 1)$. Let $p(x, y)$ be the type which says that x is greater than 0 but less than every positive real, and that y is less than rx, for any $r \in \mathbb{R}_+$, but greater than rx^{1+q}, for any $r \in \mathbb{R}$, $q \in \mathbb{Q}_+$.
It is easy to see that these conditions generate a complete consistent type.
Let f be as before, $\min(1, y/x)$.
There is no definable set, C, such that $C \in p$, f is continuous on C, and f extends continuously to $\overline{C}$.

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Basic dichotomy: cuts/noncuts

Lemma (Pillay-Steinhorn)

Let M be o-minimal, let $A = \mathrm{acl}(A)$ be a subset of M, and let $p \in S_1(A)$. Then the formulas in p of the form $x > a$, $x < a$, and $x = a$ generate p.

Definition (Marker)

For $A = \mathrm{acl}(A)$, $p \in S_1(A)$ is a cut iff it is non-algebraic and (1) there are formulas of the form $a < x$ and $x < a$ in p, and (2) for every formula of the form $a < x$ in p, there is $b > a$ such that $b < x$ is in p, and similarly for $x < a$. p is a noncut if it is non-algebraic and not a cut.

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Characterization of noncuts

- Despite their negative definition, noncuts are actually quite simple to describe. A noncut has one of the following four forms, for some $a \in A$:
- - $\{x > a\} \cup \{x < b \mid b > a, b \in A\}$
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- The first two are called, respectively, the noncut to the right (left) of a, while the last two are called, respectively, the noncut near positive (negative) infinity.
- Noncuts are the definable 1-types, definable over the (at most one)-element set containing just the "near" point.

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More preliminaries: scale

Definition

Let $M \prec N$, with every element of $N \setminus M$ definable over M. Let $p \in S_1(N)$, with p a cut over N. Let c be any realization of p. If there is an N-definable k-ary function, f, such that $f(M^k)$ is both cofinal in N below c and coinitial in N above c, we say that p is k-in scale on M. Otherwise, if there is such an f with $f(M^k)$ cofinal or coinitial, but not both, we say that p is k-near scale on M. If no such f exists, we say that p is out of scale on M.

Lemma

In the above definitions, "k" can be replaced by "1".

In view of the lemma we will drop the k and just speak of "in scale," "near scale," and "out of scale."

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Definable n-types: Marker-Steinhorn

Theorem (Marker-Steinhorn)

Let $p \in S_n(A)$. p is definable iff for some/any $c = \langle c_1, \ldots, c_n \rangle \models p$, and for $i = 1, \ldots, n$, we have $\mathrm{tp}(c_i/Ac_{<i})$ a noncut, or near scale or out of scale on A.

Scale/definability examples

Let $M = (\mathbb{R}, +, \cdot, 0, 1, <)$. Let $N = M(\epsilon)$, where ϵ is infinitesimal. For compactness of notation, let $P = \mathbb{R}_+$.

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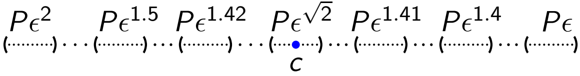

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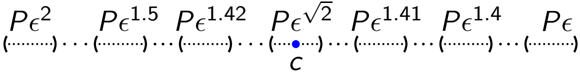

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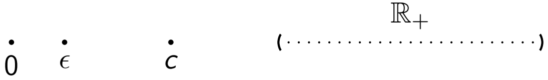
2. Let $M = (\mathbb{Q}^{\text{rcl}}, +, \cdot, 0, 1, <)$, and let $N = M(\epsilon)$. If $c \models p = \text{tp}(\pi\epsilon/N)$, then p is in scale on M since, if $f(x) = x\epsilon$, $f(M)$ is both cofinal and coinitial at c in N. Thus, $\text{tp}(\epsilon, c)$ is in scale and not definable.

More examples of scale

4. Let $M(\mathbb{R}, +, \cdot, 0, 1, <)$ and let $N = M(\epsilon)$. Let c be smaller than every real, but larger than ϵ^d, for any rational $d > 0$.



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More examples of scale

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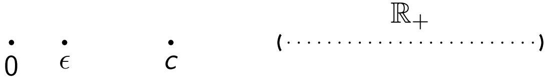

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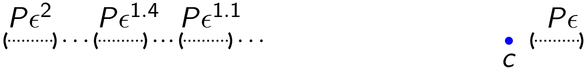

tp(c/N) is near scale on M since, if $f(x) = x\epsilon$, $f(M)$ is coinitial at c in N. tp(ϵ, c) is definable. And it is the type of our earlier counterexample.

Back to Question, and Leaping to Conclusions

If we look at our examples, we see that, in addition to (5), (2), with $\langle \epsilon, \pi\epsilon \rangle$, is easily seen to have the same failure with our question, with the same function of $\min(y/x, 1)$. So there are problems if a coordinate is near scale or in scale over the previous ones.

So perhaps each coordinate of the type being out of scale over the previous ones is the necessary criterion.

But (4) shows that we must be more careful – while $\langle \epsilon, c \rangle$ has the second coordinate near scale over the first, if we reverse the coordinates, $\langle c, \epsilon \rangle$ is just one infinitesimal followed by another, and it is not hard to show such a type cannot yield a counterexample to our question.

Since order matters as to the scale of a coordinate of a type over the previous ones, our goal is to give a presentation of the type that will enable us to examine whether one coordinate is out of scale over the previous ones without having the rug pulled out from under us via a reordering.

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Decreasing Types: The Order

Definition

Let A be a set. Define $a \prec_A b$ iff there exists $a' \in \mathrm{dcl}(aA)$ such that $a' > 0$, and $(0, a') \cap \mathrm{dcl}(bA) = \emptyset$. Define $a \sim_A b$ if $a \not\prec_A b$ and $b \not\prec_A a$. Finally, let $a \precsim_A b$ if $a \sim_A b$ or $a \prec_A b$.

This definition captures the idea that a is infinitesimal relative to b over A, or at least that some element of $\mathrm{dcl}(Aa)$ is.

Lemma

$\sim_A$ is an equivalence relation, and $\prec_A$ totally orders the $\sim_A$-classes.

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Assume that we have a fixed sequence $c = \langle c_i \rangle_{i \in I}$. Then the $\prec_i$-ordering is the $\prec_{c_{<i}}$-ordering. If we also have a fixed base set, A, then it will be the $\prec_{Ac_{<i}}$-ordering.

Definition

Let $p(x_1, \ldots, x_n) \in S_n(A)$. p is *decreasing* if, for some (any) realization, $c = \langle c_1, \ldots, c_n \rangle$ of p, $c_j \precsim_i c_i$, for $j > i$.

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Let M be an o-minimal structure expanding a real closed field. Let $p \in S_n(M)$ be a decreasing type "near" the origin. Then the following two conditions are equivalent:

1. *For $c = \langle c_1, \ldots, c_n \rangle$, some (any) realization of p, $\mathrm{tp}(c_i / c_{<i} M)$ is a noncut, or out of scale on M, for $i = 1, \ldots, n$.*
2. *For every M-definable function, f, bounded on some M-definable set in p, there is an M-definable set, C, in p, such that f is continuous on C and extends continuously to $\mathrm{cl}(C)$.*

Sketch of Backward Proof

The backward direction is fairly straightforward. Suppose that we have failure of the first condition. Then, at some coordinate, say the last one, we have some $Mc_{<n}$-definable function, g, such that $g(M)$ is near scale or in scale on M at c_n.

Consider $f = g^{-1}$ as a function of $c_{<n}$ and x. If C is any definable set containing c, we can choose $a \neq b \in M$ such that $g(a), g(b) \in C_{c_{<n}}$, and then, letting γ_1 and γ_2 be curves given by taking the pre-images of a and b under f, we get that it is impossible for f to extend continuously to the closure of C.

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Sketch of Sketch of Backward Proof

- $\langle c_1, c_2 \rangle$

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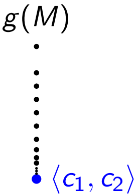

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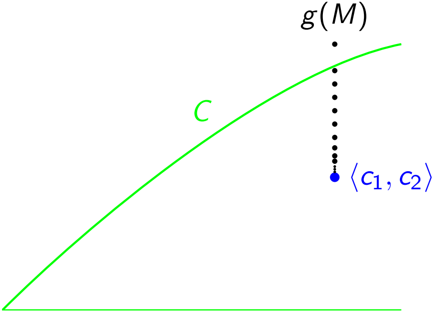

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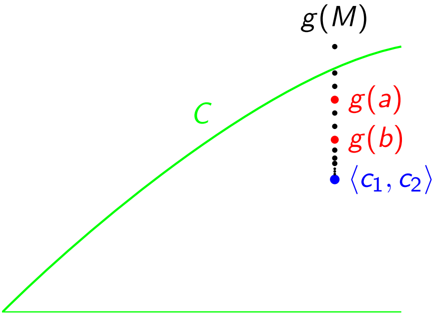

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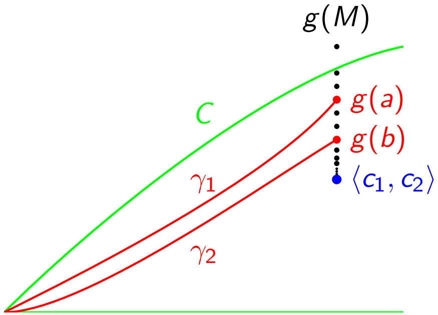

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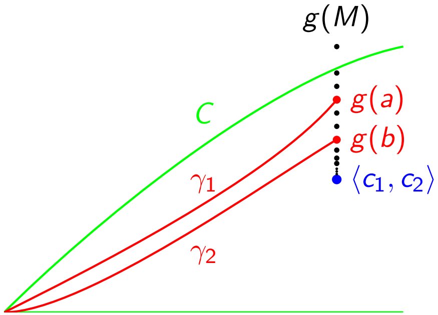

$f(x) = a$ for $x \in \gamma_1$, $f(x) = b$ for $x \in \gamma_2$.

Sketch of forward proof

For the forward direction, the proof works backwards along the coordinates of p. The auxiliary induction assumption that we use is that, when a and a' are tuples that agree through the ith coordinate, $|f(a) - f(a')|$ is bounded by a function that goes to 0 as the last coordinate that was a noncut goes to its limit.

This ensures that, when the ith coordinate is a noncut, we can continuously extend f to the closure point. To maintain the above induction assumption, we can choose a definable curve in our set, and further restrict our set so that f applied to the curve stays very "close" to the limit value of f on the curve. Then, given two points that agree on their first $i-1$ coordinates, find a point on the curve that agrees with them on their first i coordinates, and use a triangle inequality:

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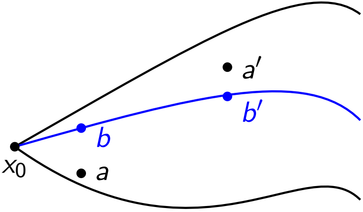
More sketchiness



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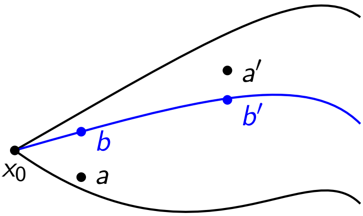

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Cut case

The noncut case is the one where any difficulties can lead to failure. The cut case is where difficulties start – where we may fail at preserving the induction assumption.

We will have to ensure that two points, a and a', that agree up to their ith coordinates, will give similar values when f is applied to them.

By doing the "opposite" of what was done in the proof of the backward direction, we can restrict to an interval that does not have any points from $f^{-1}(M)$.

From that, one can prove that two points with ith coordinates in that interval are "close enough" when f is applied to them, using results about decreasing types.

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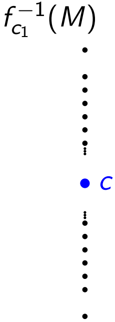

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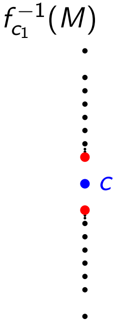

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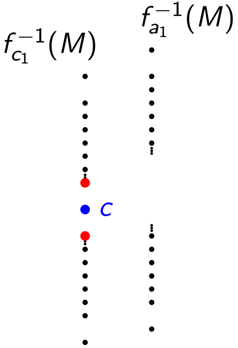

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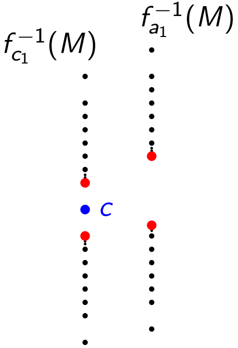

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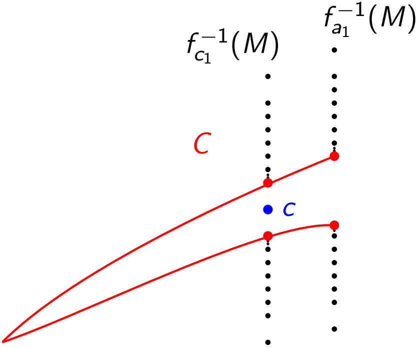

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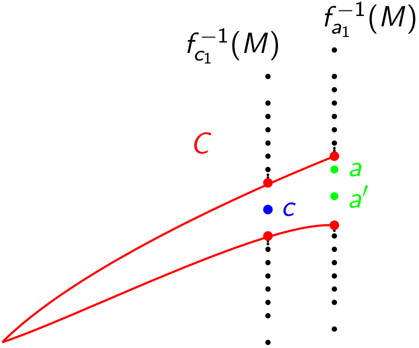

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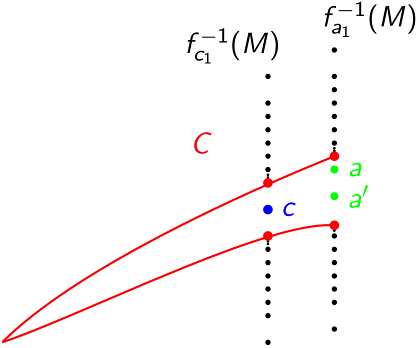

$$\mathrm{tp}(f(a)/M) = \mathrm{tp}(f(a')/M).$$

We can then show that $f(a)$ and $f(a')$ differ by a very small amount, allowing us to satisfy our induction assumption.

Conclusion

With the theorem, our original case of a curve is resolved, by taking the curve's limit type.

While in this case, we were restricted from taking types that were interdefinable with our original, in circumstances where one can (for example, when examining definability), decreasing types allow for tighter results, since all near scale and in scale types can be removed – even our example of (5) disappears.

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