Scale, decreasing types, and extending functions continuously in o-minimal theories

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Extending Functions to Closures[1]

Let γ be a curve in M^n with one endpoint the origin, and let f be an M-definable bounded n-ary function. Can we find an initial segment of γ and a definable set containing that initial segment on which f is continuous, or extends continuously?

Note that we can certainly find a definable set containing $\gamma \setminus \{0\}$ on which f is continuous. The difficulty is in extending f continuously to 0, which is equivalent to extending f continuously to the closure of the definable set.

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Definable γ works

Example

Let $f(x, y) = \min(1, y/x)$, and let γ be any definable curve in the first quadrant with left endpoint 0.

We can take a pair of parabolas whose derivatives at 0 are the same as the curve's at 0, giving us a cell on which f extends continuously to the closure.

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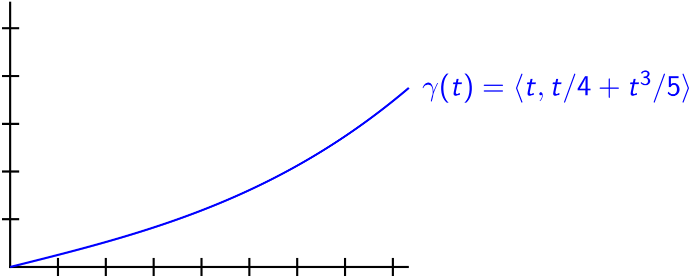

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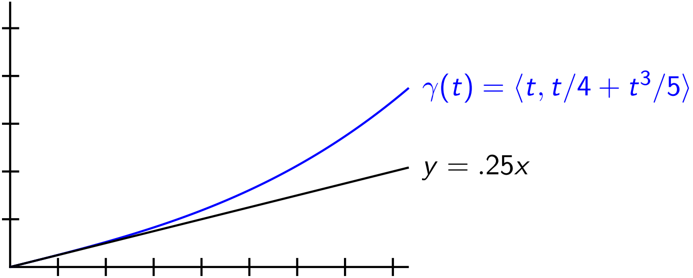

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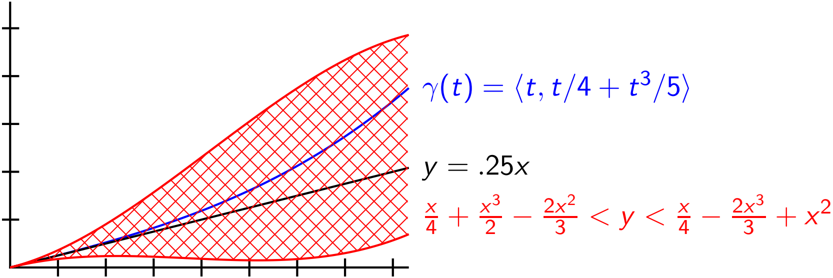

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What about the question for non-definable curves? Given a (non-definable) curve, can we find a set on which the function is continuous, which contains the curve, and on whose closure the function extends continuously?

Restricting to Good Curves

- Clearly, we must impose some conditions on our curves.
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Comparing Curves

Definition

Let f and g be unary functions (not necessarily definable), each of whose domains includes some positive neighborhood of 0. f and g are comparable if, for some $s > 0$, one of a) for all $t \in (0, s)$, $f(t) < g(t)$; b) for all $t \in (0, s)$, $f(t) = g(t)$; or c) for all $t \in (0, s)$, $f(t) > g(t)$.

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Ordered Curves

Definition

Let M be any o-minimal structure, and let $\gamma = \langle \gamma_1, \ldots, \gamma_n \rangle$ be a (not necessarily definable) curve in M^n. Say that γ is *ordered* if, for $i = 2, \ldots, n$, γ_i is comparable to every function in the set

$$\{f(\gamma_{i_1}(t), \ldots, \gamma_{i_k}(t)) \mid f \text{ is an } M\text{-definable } k\text{-ary function}, i_1, \ldots, i_k < i\},$$

and γ_1 is comparable to every M-definable function of t.

Note that whether or not γ is ordered does not depend on the ordering of the coordinates of γ.

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Ordered Failure

- Let $M = (\mathbb{R}, +, \cdot, <, 0, 1)$. Let $f(x, y)$ be $\min(1, y/x)$, and let $\gamma(t) = \langle t, -t/\ln t\rangle$, so γ is undefinable in M. Note, though, that since γ is definable in the o-minimal expansion of M, $(\mathbb{R}, +, \cdot, <, \exp)$, γ is certainly ordered.
- $-1/\ln t$ goes to 0, but it is also greater than t^d, for any $d > 0$, for sufficiently small t. Thus, $-t/\ln t$ is greater than t^{1+d}, but less than at, for every $a \in \mathbb{R}_+$.
- It is not hard to see that any definable set in $(\mathbb{R}, +, \cdot, <, 0, 1)$ that contains γ must contain the curve $\langle t, at\rangle$, for some real positive a. But then it will also contain the curve $\langle t, at/2\rangle$.
- f cannot be continuously extended onto this set's closure, because the limit along the first curve is a and the second curve is $a/2$.

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Why Did γ Fail?

The failure of γ can be seen as coming from the fact that we could not squeeze γ sufficiently to keep out incompatible curves, because the gap between a linear function and a higher-power function is too great. To more closely analyze this, we can abstract out the "type" of γ.

The Limit Type of a Curve

Lemma

Let $\gamma = \langle \gamma_1, \ldots, \gamma_k \rangle$ *be an ordered curve. Let* $\gamma(t)$ *denote the sequence* $\langle \gamma_1(t), \ldots, \gamma_k(t) \rangle \in M^k$*, for* $t \in M$*. Then* $\lim_{t \to 0^+} \mathrm{tp}(\gamma(t)/M)$ *exists, in the following sense: for each formula* $\psi(x_1, \ldots, x_k)$ *in* M*, there is some* $s > 0$ *such that either* $\psi(\gamma(t))$ *holds for all* $t \in (0, s)$*, or* $\neg\psi(\gamma(t))$ *holds for all* $t \in (0, s)$*.*

Curve Limit Type Determines Definable Set Membership

Definition

With γ as above, let $\operatorname{tp}(\gamma/M)$ denote $\lim_{t\to 0^+} \operatorname{tp}(\gamma(t)/M)$. We can then talk about the type of γ_i over $\gamma_{<i}M$.

Fact

Let γ be an ordered curve. Then, for any definable C, there exists an $s > 0$ such that $\gamma((0,s)) \subseteq C$ if and only if $C \in \operatorname{tp}(\gamma/M)$.

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Back to Bad γ

We now return to $\gamma(t) = \langle t, -t/\ln t\rangle$, and examine $\mathrm{tp}(\gamma)$.

- For every $r > 0 \in \mathbb{R}_+$, $x_1 < r$ is in $\mathrm{tp}(\gamma)$.
- For every $r \in \mathbb{R}_+$, $x_2 < rx_1$ is in $\mathrm{tp}(\gamma)$.
- For every $r \in \mathbb{R}$, $q \in \mathbb{Q}_+$, $x_2 > rx_1^{1+q}$ is in $\mathrm{tp}(\gamma)$.

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Failure of Continuity Extension for a Type

Since we have equivalence of definable set membership for curves and their types, we can rephrase our failure with γ as follows:

Example

Take our model to be $(\mathbb{R}, +, \cdot, <, 0, 1)$. Let $p(x, y)$ be the type which says that x is greater than 0 but less than every real, and that y is less than rx, for any $r \in \mathbb{R}_+$, but greater than rx^{1+q}, for any $r \in \mathbb{R}$, $q \in \mathbb{Q}_+$. It is easy to see that these conditions generate a complete consistent type. Let f be as before, $\min(1, y/x)$.
There is no definable set, C, such that $C \in p$, f is continuous on C, and f extends continuously to $\overline{C}$.

If we let $\langle c_1, c_2 \rangle \models p$, the problem here is that the pre-images of elements of $\mathbb{R}$ under $f(c_1, -)$ are coinitial at c_2 in $\mathbb{R}(c_1)$.

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Scale

Definition (∼Marker-Steinhorn)

Let $A \subset B$, and $p \in S_1(B)$, with p a cut over B. Let c be any realization of p. If there is a B-definable unary function, f, such that $f(A)$ is both cofinal in B below c and coinitial in B above c, we say that p is in scale on A. Otherwise, if there is such an f with $f(A)$ cofinal or coinitial, but not both, we say that p is near scale on A. If no such f exists, we say that p is out of scale on A.

Scale examples

Let $M = (\mathbb{R}, +, \cdot, 0, 1, <)$. Let $N = M(\epsilon)$, where ϵ is infinitesimal. For compactness of notation, let $P = \mathbb{R}_+$.

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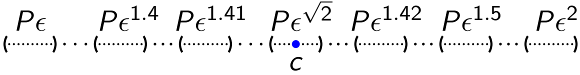

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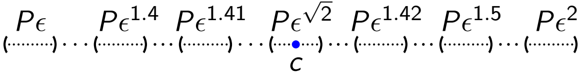

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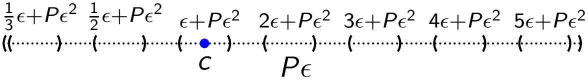
2. If $c \models p = \text{tp}(\sum_{i=1}^{\infty} \epsilon^i/N)$, then



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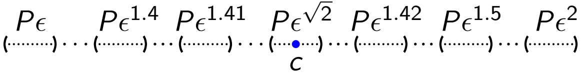

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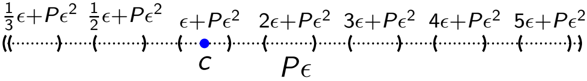

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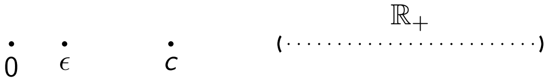
3. Let $M = (\mathbb{Q}^{\mathrm{rcl}}, +, \cdot, 0, 1, <)$, and let $N = M(\epsilon)$. If $c \models p = \mathrm{tp}(\pi\epsilon/N)$, then p is in scale on M since, if $f(x) = x\epsilon$, $f(M)$ is both cofinal and coinitial at c in N.

More examples of scale

4. Let $M(\mathbb{R}, +, \cdot, 0, 1, <)$ and let $N = M(\epsilon)$. Let c be smaller than every real, but larger than ϵ^d, for any rational $d > 0$.



tp(c/N) is near scale on M since, if $f(x) = x$, $f(M)$ is coinitial at c in N. However, note that, if we take $N' = M(c)$, then ϵ is a noncut over N', so the scale issue does not arise.

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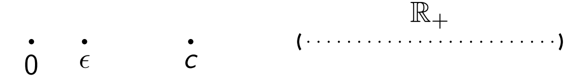

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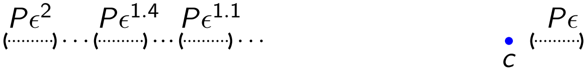
5. Let $M = (\mathbb{R}, +, \cdot, 0, 1, <)$ and $N = M(\epsilon)$, and let c be smaller than $r\epsilon$ for $r \in \mathbb{R}_+$, but larger than ϵ^q for $q \in \mathbb{Q}_{>1}$.



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Leaping to Conclusions

If we look at our examples, we see that, in addition to 5, 3, with $\langle \epsilon, \pi\epsilon \rangle$, is easily seen to have the same failure, with the same function of $\min(y/x, 1)$. So there are problems if a coordinate is near scale or in scale over the previous ones.

So perhaps each coordinate of the type being out of scale over the previous ones is the necessary criterion.

But 4 shows that we must be more careful – while $\langle \epsilon, c \rangle$ has the second coordinate near scale over the first, if we reverse the coordinates, $\langle c, \epsilon \rangle$ is just one infinitesimal followed by another, and it is not hard to show such a type cannot yield a counterexample.

Since order matters as to the scale of a coordinate of a type over the previous ones, our goal is to give a presentation of the type that will enable us to examine whether one coordinate is out of scale over the previous ones without having the rug pulled out from under us via a reordering.

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Decreasing Types: The Order

Definition

Let A be a set. Define $a \prec_A b$ iff there exists $a' \in \mathrm{dcl}(aA)$ such that $a' > 0$, and $(0, a') \cap \mathrm{dcl}(bA) = \emptyset$. Define $a \sim_A b$ if $a \not\prec_A b$ and $b \not\prec_A a$. Finally, let $a \precsim_A b$ if $a \sim_A b$ or $a \prec_A b$.

This definition captures the idea that a is infinitesimal relative to b over A, or at least that some element of $\mathrm{dcl}(Aa)$ is.

Lemma

$\sim_A$ is an equivalence relation, and $\prec_A$ totally orders the $\sim_A$-classes.

Definition

Let A be a set. Define $a \prec_A b$ iff there exists $a' \in \mathrm{dcl}(aA)$ such that $a' > 0$, and $(0, a') \cap \mathrm{dcl}(bA) = \emptyset$. Define $a \sim_A b$ if $a \not\prec_A b$ and $b \not\prec_A a$. Finally, let $a \precsim_A b$ if $a \sim_A b$ or $a \prec_A b$.

This definition captures the idea that a is infinitesimal relative to b over A, or at least that some element of $\mathrm{dcl}(Aa)$ is.

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Decreasing Types: The Order

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Decreasing Types: Definition

Definition

Assume that we have a fixed sequence $c = \langle c_i \rangle_{i \in I}$. Then the $\prec_i$-ordering is the $\prec_{c_{<i}}$-ordering. If we also have a fixed base set, A, then it will be the $\prec_{Ac_{<i}}$-ordering.

Definition

Let $p(x_1, \ldots, x_n) \in S_n(A)$. p is *decreasing* if, for some (any) realization, $c = \langle c_1, \ldots, c_n \rangle$ of p, $c_j \precsim_i c_i$, for $j > i$.

Lemma

Any n-type can have its coordinates reordered so that it is decreasing.

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Let M be an o-minimal structure expanding a real closed field. Let $p \in S_n(M)$ be a decreasing type "near" the origin. Then the following two conditions are equivalent:

1. *For $c = \langle c_1, \dots, c_n \rangle$, some (any) realization of p, $\mathrm{tp}(c_i / c_{<i} M)$ is a noncut, or out of scale on M, for $i = 1, \dots, n$.*
2. *For every M-definable function, f, bounded on some M-definable set in p, there is an M-definable set, C, in p, such that f is continuous on C and extends continuously to $\mathrm{cl}(C)$.*

Sketch of Backward Proof

The backward direction is fairly straightforward. Suppose that we have failure of the first condition. Then, at some coordinate, say the last one, we have some $Mc_{<n}$-definable function, g, such that $g(M)$ is near scale or in scale on M at c_n.

Consider $f = g^{-1}$ as a function of $c_{<n}$ and x. If C is any definable set containing c, we can choose $a \neq b \in M$ such that $g(a), g(b) \in C_{c_{<n}}$, and then, letting γ_1 and γ_2 be curves given by taking the pre-images of a and b under f, we get that it is impossible for f to extend continuously to the closure of C.

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Sketch of Sketch of Backward Proof

- $\langle c_1, c_2 \rangle$

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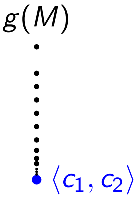

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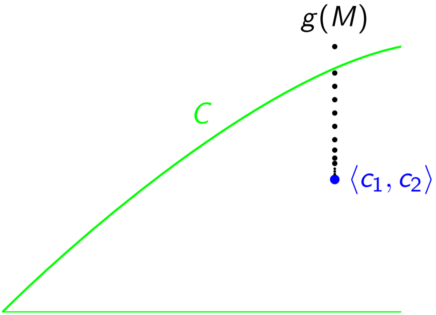

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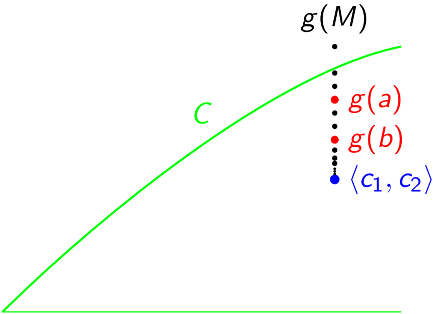

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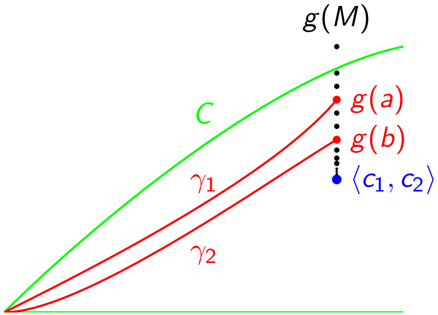

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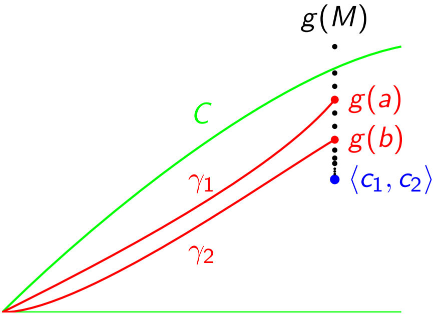

$f(x) = a$ for $x \in \gamma_1$, $f(x) = b$ for $x \in \gamma_2$.

Conclusion

With the theorem, our original case of a curve is resolved, by taking the curve's limit type.

While in this case, we were restricted from taking types that were interdefinable with our original, in circumstances where one can (for example, when examining definability), decreasing types allow for tighter results, since all near scale and in scale types can be removed – even our example of 5 disappears.

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