Partial orders in tame ordered structures

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Question

- J. Truss asked whether any definable partial order in an o-minimal structure could be definably extended to a linear order.
- We will positively answer a generalization of this question, by describing several classes of ordered structures that definably extend their definable partial orders.
- These structures can be thought of as possessing a "definable" order extension principle – in these structures, the "order extension principle" of ZFC holds definably. Formally:

Definition

Let M be a structure. Say that M *has the order extension principle* (*has OE*) if, for any M-definable partial order $(P, \prec)$, there is an M-definable linear order $\prec'$ that totally orders P and such that $x \prec y \Rightarrow x \prec' y$.

Examples of structures with OE

In this talk, we will prove that the following structures have OE:

1. All well-ordered structures.
2. All (weakly) o-minimal structures (every definable 1-dimensional set is a finite union of points and convex sets).
3. All (weakly-)quasi-o-minimal structures.

- Prior to our work, the only results in this direction were when the partial order was 1-dimensional (just a subset of M).
- MacPherson and Steinhorn did the case when M was o-minimal.
- Felgner and Truss did the case when M was well-ordered, essentially by the same method as our proof.

The key easy step

- Our work hinges on an easy observation: that any family of sets induces a partial order on its parameter set.
- Let $\mathcal{V} = \{V(x) : x \in A\}$ be any family of sets, parameterized by A.
- Let $\prec_{\mathcal{V}}$ be the partial order on A given by the relation $x \prec_{\mathcal{V}} y$ if and only if $V(x) \subsetneq V(y)$.

Definition

Let $(P, \prec)$ be a partial order. Let $L(x) = \{y \in P : y \prec x\}$ for $x \in P$ – the "lower cone" of x.

- Let $\mathcal{V} = \{L(x) : x \in P\}$. Then $\prec_{\mathcal{V}}$ is a partial order on P.
- Note that if $x \prec y$, then by transitivity and the fact that $x \in L(y) \setminus L(x)$, we have $x \prec_{\mathcal{V}} y$, so $\prec_{\mathcal{V}}$ is a partial order on P extending $\prec$.
- Thus, if we can linearly extend the partial order $\prec_{\mathcal{V}}$ for any definable family $\mathcal{V}$, we can linearly extend any partial order.

Well-ordered structures

Theorem

Let M be a well-ordered structure. Then M has OE.

- Let A be the parameter set for $\mathcal{V} = \{V(x) : x \in A\}$, a definable family of sets in M^n for some $n \geq 0$. We consider the case $n = 1$. The general case is similar.
- For $x, y \in A$, let $B(x, y) = V(x) \triangle V(y)$. Since M is well-ordered, there is a least element of $B(x, y)$. Then for $x, y \in A$, let $x \prec y$ if $t \in V(y)$ (so $t \notin V(x)$).
- If x and y are still unordered, then $V(x) = V(y)$. Order x and y lexicographically.

Dimension n

- For $t \in M$ and any set $X \subseteq M^n$, let $X_t = \{y : \langle t, y\rangle \in X\}$, the fiber of X over t.
- For higher dimensions, we use the fact that for any $t \in M$, we can consider the family $\mathcal{V}_t = \{V(x)_t : x \in A\}$.
- This induces a partial order $\prec_t$ on A.
- The collection $\mathcal{V}_t$ is a family of $(n-1)$-dimensional sets and so, by induction, we may extend each $\prec_t$ to a linear order on A, uniformly in t.
- Instead of letting $B(x, y) = V(x) \triangle V(y)$, we set $B(x, y) = \{t : V(x)_t \neq V(y)_t\}$. Then we let $x \prec y$ if $x \prec_t y$ for t the least element of $B(x, y)$.

The general case

The previous proof gives the principle for subsequent proofs: if there is some consistent way to pick out a particular part of $B(x, y)$, for which each $\prec_t$ gives the same answer about x and y, then we can use that answer to order x and y.

Theorem (R., Steinhorn)

Let M be an ordered structure such that, for any definable $A, C \subseteq M$, there is some initial segment of A either contained in or disjoint from C. Then M has OE.

Proof.

- As before, we restrict to the 1-dimensional case for simplicity.
- The proof proceeds as in the well-ordered case until we have $B(x, y) = V(x) \triangle V(y)$.
- Consider the definable set $\{t : t \in V(y) \setminus V(x)\}$. By hypothesis, this set either contains or is disjoint from an initial segment of $B(x, y)$.
- If it contains an initial segment of $B(x, y)$, then set $x \prec y$. Otherwise, let $y \prec x$.
- It is then routine to verify that this yields a nearly-total order, which is completed lexicographically.

$\square$

Consequences of Theorem

Theorem

If M is an ordered structure such that for any definable $A, C \subseteq M$, C contains or is disjoint from an initial segment of A, then M has OE.

- The theorem immediately implies our results on well-ordered, o-minimal, and weakly o-minimal structures.
- Note that while the hypothesis on M in the theorem is first-order, the properties of being well-ordered or weakly o-minimal are not first-order.
- Thus, if *some* model of the theory of M is weakly o-minimal or well-ordered, then M satisfies the requisite hypothesis.

Extending the proof

- As referred to before, if there is some consistent way to pick out a particular part of $B(x, y)$, for which each $\prec_t$ gives the same answer about x and y, then we can use that answer to order x and y.
- We thus describe a class of structures for which a more intricate model-theoretic argument works.

Confusing property

Recall that a structure is *ω-saturated* if any type over finitely many element is realized in the structure itself.

Definition

Say that an ω-saturated ordered structure M has $(\ddagger)$ if for any complete type $p \in S_1(\emptyset)$ and any definable sets $A, C \subseteq M$, the set $p(M) \cap A$ has an initial segment either disjoint from or contained in C.

- This is a natural generalization of the previous property we looked at.
- Instead of looking at the whole structure when we intersect sets, we restrict to a $\emptyset$-definable type.
- This avoids problems caused by things like $\emptyset$-definable predicates.

Lemma

If M has $(\ddagger)$, then, given A and C, we may actually replace the type p in the statement of $(\ddagger)$ by some formula $\varphi \in p$. Thus some initial segment of $\varphi(M) \cap A$ is contained in or disjoint from C. Moreover, φ is independent of the parameters used to define A, C.

The lemma comes from a straightforward use of compactness, and allows us to replace types by formulas.

Theorem (R., Steinhorn)

Let M be an ω-saturated ordered structure with $(\ddagger)$. Then M has OE.

The proof proceeds as before, but the definition of the order in terms of $B(x, y)$ will be considerably more complicated, due to multiple applications of compactness.

- Let $x, y \in A$. As before, we consider the one-dimensional case.
- We want to look at $V(y) \setminus V(x)$ on an "initial segment" of $B(x, y)$.
- However, the set $V(y) \setminus V(x)$ may not behave nicely on an initial segment of $B(x, y)$, so we must consider it on types near the lower boundary of $B(x, y)$.
- Let $C(x, y)$ be the upward closure of $B(x, y)$.
- Let P be the set of all types over $\emptyset$ that have realizations coinitial in $C(x, y)$.
- For each type $p \in P$, there is some formula φ_p such that for t in an initial segment of $\varphi_p(M) \cap C(x, y)$, the statements $t \in V(x)$ and $t \in V(y)$ have constant truth value.
- There is some type $p \in P$ such that $t \in V(x) \triangle V(y)$ for t coinitial in $p(M) \cap C(x, y)$.
- For this p, we know that on an initial segment of φ_p, we have (without loss of generality) $t \in V(y) \setminus V(x)$.
- Since there are such p and φ_p for every choice of x', y' such that $C(x', y') = C(x, y)$, there must be finitely many choices for φ_p.
- Applying compactness again, this time allowing x and y to vary so that $C(x, y)$ changes, we obtain finitely many formulas $\varphi_1, \ldots, \varphi_m$ such that for any $x, y \in A$ and each $i \leq m$, either:
  1. On an initial segment of $\varphi_i(M) \cap C(x, y)$ we have (say) $t \in V(y) \setminus V(x)$.
  2. On an initial segment of $\varphi_i(M) \cap C(x, y)$ we have $t \in V(x) \iff t \in V(y)$.
- Now we can set $x \prec y$ if and only if on the first $i \leq m$ such that (2) fails, we have $t \in V(y) \setminus V(x)$ on an initial segment of $\varphi_i(M) \cap C(x, y)$.
- Verification that this is a partial order extending the original is routine, since in some sense it is a "lexicographic" order based on the behavior on each φ_i.

Consequences

- This theorem most directly deals with quasi-o-minimal structures: ordered structures in which every definable set is (uniformly) a finite Boolean combination of points, intervals, and $\emptyset$-definable sets.
- We can also weaken "interval" to "convex set," obtaining weakly-quasi-o-minimal structures.
- There is a dp-minimal ordered structure that fails the hypothesis of the theorem.
- But I don't know of any example of a totally ordered structure that does not have OE.