Interpretable groups are definable

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Definitions

- An *o-minimal* structure M is a linearly ordered structure in which every first-order definable subset of M is a finite union of points and intervals. The reals as an ordered field and the rationals as an ordered group are both examples.
- We will only consider densely ordered o-minimal structures.
- A structure $(G, \ldots)$ is *interpretable* in M if there is a definable set $X \subseteq M^k$ and definable equivalence relation E such that G is isomorphic to X/E and all the structure on G is definable on X/E in M. The structure M^{eq} contains all interpretable sets in M.
- Many o-minimal structures have *elimination of imaginaries*: every interpretable set is definably isomorphic to a definable one. In fact, they often have definable choice.
- This follows from cell decomposition in the presence of a group structure. Each equivalence class can be taken to be a union of cells, and the structure can uniformly pick a unique element in each cell.

Local Properties

- The assumption of group structure is not so strange, because by the Trichotomy Theorem, every point in an o-minimal structure is either "trivial," lies in a definable local group, or lies in a definable real closed field.
- "Trivial" means that there are no definable monotonic binary functions in a neighborhood. A "local group" can be thought of as the restriction of a topological group to a neighborhood around 0, so addition is not always defined, if it would go outside the neighborhood.
- However, it is certainly possible that a structure can have a definable local group around every point, and yet not have a definable global group, or even admit the structure of a global group.

Groups

- Besides global group structure and local groups, o-minimal structures can also have general definable or interpretable groups.
- These groups live in some cartesian power of the structure, and need not, a priori, have anything to do with any underlying group in the structure.
- Examples include the circle group S_1 and general linear group $GL_n(R)$ on the definable side, and $PGL_n(R)$ on the interpretable side.
- There has been much work about definable groups, most prominently in the proof of Pillay's Conjecture, that every definable group, after a quotient by the connected component G^{00} is isomorphic to a Lie group of the appropriate dimension.
- However, little was known about interpretable groups.

Result

Theorem

Let G be an interpretable group in a dense o-minimal structure. Then G is definably isomorphic to a definable group that is a subset of a cartesian product of one-dimensional definable groups.

- Note: the definable isomorphism may require more parameters than those used to define G.
- When M expands a group, the theorem is trivial. Thus, the principle of the proof is to use the existence of the group G to accomplish what the group on M would normally do.
- When M does not expand a group, the conclusion was unknown even for definable groups.

Strategy: Topology

Our strategy requires a number of approaches on different aspects of o-minimality.

- As with definable groups, we endow G with a group topology with a definable basis. In this process, we essentially turn G into a manifold. While the manifold does not have a finite atlas, it does yield a finite number of "large" sets, through which we can deduce many of the usual properties of definable groups.
- Using standard topological group decompositions, we can separate into the definably simple and definably compact cases for G.
- When G is definably simple (non-abelian with no definable nontrivial normal subgroup) and definably connected, we repeat the proof of Peterzil-Pillay-Starchenko using our group topology and manifold structure.
- The techniques of [PPS] embed G into $GL(n, R)$ for some definable real closed field R. Since $GL(n, R)$ is a definable group, this finishes the theorem for definably simple groups.

Strategy: Getting One-Dimensional Sets

- When G is definably compact, we use a strategy similar to Edmundo's in the case of solvable groups to obtain strong definable choice for M^{eq}-definable subsets of G.
- Strong definable choice means that for any definable family $\{X_t \subseteq G : t \in T\}$ with $T \subseteq M^{\mathrm{eq}}$, there is a definable function $f : T \to G$ such that $f(t) \in X_t$ and $f(t) = f(s)$ if $X_t = X_s$.
- A general result: for interpretable X/E, we can take $X \subseteq I_1 \times \cdots \times I_k$, with each interval I_j the image of X/E under a definable map f_j.
- Applying this result to definably compact G and using strong definable choice on the sets given by the preimages of the f_j's, we have one-dimensional subsets of G.

Strategy: Turning One-Dimensional Sets Into Groups

- We prove a general result that any one-dimensional equivalence relation can be eliminated – that is, if $\dim(X/E) = 1$, then X/E is in definable bijection with a one-dimensional definable set.
- Thus, any one-dimensional subset of G is in definable bijection with a one-dimensional subset of M.
- We want these one-dimensional subsets to be embedded in definable groups, so we can definably choose representatives of each equivalence class in G.
- We then prove that if $f : I \times J \to M$ is a definable function, monotonic in both coordinates, then either I or J can be definably embedded in a definable one-dimensional group.
- Applying to the group operation on I_j yields the desired result.

What to Expect

In this talk, I will:

- show where the topology comes from;
- give the proof that one-dimensional quotients can be eliminated;
- give some idea why if $f : I \times J \to M$ is a definable function monotonic in both coordinates, then either I or J can be definably embedded in a definable one-dimensional group.

We do not have a finite atlas (yet) on G with this topology. However, what we have is not too bad:

Proposition

There are finitely many t-open definable sets $W_1, \ldots, W_k$ whose union covers G. Each W_i is the (non-injective!) image of $\mathcal{U}_0$, where $\mathcal{U}_0$ is a finite disjoint union of definable open subsets of various M^{r_i}'s.

This implies that every definable subset of G has finitely many definably connected components, and thus that many properties of definable groups in o-minimal structures still hold.
In particular, this is enough for the definably simple non-abelian case, with [PPS]'s arguments.

Topology

- We can modify our underlying set X and equivalence relation E so that after a partition, all equivalence classes have the same homeomorphism type, and the base set U is open in its ambient space.
- We suppose that each equivalence class is open in the first d coordinates. Then for each $x \in \pi_{\leq d}(U) \subseteq M^n$, the fiber of U above x has a single representative in each E-class.
- For $u \in U$, let $U(u)$ be the fiber of U above $\pi_{\leq d}(u)$.
- Let $u = \langle x', x'' \rangle$ be a generic element of U, and let $\mathcal{V}$ be a definable basis of neighborhoods of x'', all contained in $U(u)$. Then the family $\mathcal{B} = \{g\mathcal{V} : g \in G\}$ is a basis for a topology (t-topology) making G into a topological group.
- The t-topology makes G into a topological group because it comes from the usual order topology, so there is a canonical homeomorphism between a neighborhood V of x'' in $U(u)$ and a t-neighborhood of u.

One-dimensional interpretable sets

- The proof for definably compact groups goes by first showing that definably compact groups have strong definable choice.
- This then allows us to definably pick one-dimensional *interpretable* sets in the group G, into whose cartesian product we can suppose that G is embedded.
- Thus, if we can show that these one-dimensional interpretable sets are actually definable and embeddable in one-dimensional groups, we will be done.

Theorem

Let $T \subset M^{eq}$ have dimension 1. Then there exists a definable injective map $f : T \to M^m$ for some m.

We consider $\{X_t : t \in T\}$ a definable family, with $T \subset M^{\text{eq}}$ and $\dim T = 1$, and show that the desired map exists for this T, by induction on the ambient space of the X_t's. Then we are done by considering $\{[t] : t \in T\}$.

- We perform o-minimal tricks to make all the X_t's cells in M^k of the same dimension r. We go by induction on (k, r).
- If $r = k$, then each X_t is uniquely determined by its "boundary cells," and we are done by induction. So we can take $r = k - 1$.
- There are two kinds of points in the X_t's – those that belong to only finitely many X_t, and the others. We partition each X_t into these two sets, X_t^0 and X_t'.
- The union of all X_t' has dimension less than k, by straightforward dimension arguments, so it is done by induction.

- Further partitioning X_t^0, we can suppose that it is the graph of a function f_t on a cell C_t, with distinct X_t^0's disjoint.
- By induction, we have the desired function for the family $\{C_t : t \in T'\}$, where T' is T modulo the equivalence relation $C_s = C_t$. So we need to separate out X_t's projecting to the same C_t.
- For each C_t, if only finitely many X_t^0 project onto C_t, then we can take care of them.
- If infinitely many X_t^0 project onto C_t, then since $\dim T = 1$, there are only finitely many such C_t. For each one, we can fix $\bar{a} \in C_t$, and define $g(t) = f_t(\bar{a})$.
- (This step fails in higher dimension, since we would have to pick infinitely many such points.)

Group-intervals

- We have now reduced the problem of definably compact G to showing that one-dimensional definable subsets of G embed in definable groups.
- Every point of such a set is non-trivial (has a definable local group) around it. But we need a local group that contains the whole set, up to a finite partition.

Definition

Let I be a *gp-short interval* if after a finite partition, it can be definably endowed with the structure of a group chunk, with 0 either an endpoint of I or in I.

Lemma

Let $\{I_t : t \in T\}$ be a definable family of gp-short intervals, all with the same left endpoint. Then $\bigcup_t I_t$ is a gp-short interval.

No demands are made on how the group chunks on I, I_t are defined.

Proof:

- Let $(a, b) = \bigcup_t I_t$. We replace M by (a, b) with all the induced structure on (a, b).
- If we can find $c \in (a, b)$ such that (c, b) is a group interval, then we will be done, since some I_t contains (a, c).
- If there is $c \in (a, b)$ with a definable injection from (c, b) to (d, e) for some $a < d < e < b$, again we are done.
- Thus, we may assume that there are no such maps for any c, and thus that our structure has no "poles," treating b like ∞.
- We pick a nonstandard $c < b$ in an elementary extension N of M. The interval (a, c) is gp-short, so there is a group operation $+_G$ on it.

- The fact that there are no poles means that the left convex hull of M in N, $M' = \{x \in N : \exists d \in M(x < d)\}$, is an elementary substructure of N.
- The type of any element of N over M' is definable, since any element of $N \setminus M'$ is infinitely large.
- Then by the Marker-Steinhorn theorem, the type of any tuple of elements of N over M' is definable.
- Thus, the trace of any N-definable set in M' is M'-definable. It is straightforward that the trace of $+_G$ on M' gives a local group on all of M', so M' itself defines a total local group.
- Since this property is first-order, $M = (a, b)$ also defines a total local group.

- The standard machinery of local modularity gives a group operation around $x_0 \in I$ by $x_1 + x_2 = x_3 \iff f_{x_2}^{-1} f_{x_3} = f_{x_0}^{-1} f_{x_1}$.
- This operation is valid whenever the intervals (x_1, x_0) and (x_2, x_0) are gp-short.
- But being gp-short is not a definable property, so the operation "spills over" onto a longer interval, which is necessarily gp-long, contradiction.

Everything Interesting is gp-short

Theorem

Let I, J be intervals, and $f : I \times J \to M$ a definable function strictly monotone in both variables. Then at least one of I or J is gp-short.

Some steps on the way to the proof:

- If $f : I_1 \times \ldots \times I_k \to J$ is definable with J gp-short and all I_i gp-long, then f is constant at every generic point.
- If $f : I_1 \times I_2 \times J \to M$ is definable with J gp-short but I_1, I_2 gp-long, then for generic $a \in I_1 \times I_2$, the function $f(a, -)$ is determined (up to finite) by $f(a, d)$ for any generic d.
- If $f : I_1 \times I_2 \times I_3 \to M$ is definable with I_1, I_2, I_3 gp-long, then we can partition I_1, I_2, I_3 so that the functions $f(a, -)$ and $f(b, -)$ on I_3 are identical if they ever have the same value.
- So families of functions parameterized by gp-long intervals are one-dimensional, i.e., locally modular.

Applying the Theorem

- By an argument, if $h : I_1 \times \ldots \times I_{k+1} \to M^k$ is a definable map injective in each coordinate separately, then at least one of $I_1, \ldots, I_{k+1}$ is gp-short.
- Let I be a one-dimensional set definable in G. Let $f_i : I^i \to G$ be defined by $f_i(x_1, \ldots, x_i) = x_1 \cdots x_i$.
- Take $k \geq 1$ maximal such that f_k is injective on B, some cartesian product of gp-long intervals in I^k.
- We find a generic $k+1$-tuple $\langle a_1, \ldots, a_{k+1} \rangle \in I^{k+1}$ and a box B' around it such that $f_{k+1}(B')$ is contained in $f_k(B) \cdot a_{k+1}$.
- This is enough, because then we are mapping a $k+1$-dimensional set injectively in each coordinate into a k-dimensional set.