# Interpretable groups are definable

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## Definitions

- An *o-minimal* structure $M$ is a linearly ordered structure in which every first-order definable subset of $M$ is a finite union of points and intervals (due to Pillay, Steinhorn).
- We will only consider densely ordered o-minimal structures.
- A structure $(G, \dots)$ is *interpretable* in $M$ if there is a definable set $X \subseteq M^k$ and definable equivalence relation $E$ such that $G$ is isomorphic to $X/E$ and all the structure on $G$ is definable on $X/E$ in $M$.

## Result

**Theorem**

*Let $G$ be an interpretable group in an arbitrary dense o-minimal structure $M$. Then $G$ is definably isomorphic to a definable group that is a subset of a cartesian product of one-dimensional definable groups.*

- Note: the definable isomorphism may require more parameters than those used to define $G$.
- When $M$ expands a group, the theorem is trivial. Thus, the principle of the proof is to use the existence of the group $G$ to accomplish what the group on $M$ would normally do.

## Strategy

Our strategy requires a number of approaches on different aspects of o-minimality.

- As with definable groups, we endow $G$ with a group topology with a definable basis.
- When $G$ is definably simple (non-abelian with no definable nontrivial normal subgroup) and definably connected, we can repeat the proof of Peterzil-Pillay-Starchenko using this group topology.
- The proof of [PPS] yields an embedding of $G$ into $GL(n, R)$ for some definable real closed field $R$. Since $GL(n, R)$ is a definable group, this finishes the theorem.
- When $G$ is definably compact, we use a strategy similar to Edmundo's in the case of solvable groups to obtain strong definable choice for $M^{eq}$-definable subsets of $G$.
- Strong definable choice means that for any definable family $\{X_t \subseteq G : t \in T\}$, there is a definable function $f : T \to G$ such that $f(t) \in X_t$ and $f(t) = f(s)$ if $X_t = X_s$.

- A general result: for interpretable $X/E$, we can take $X \subseteq I_1 \times \cdots \times I_k$, with each interval $I_j$ the image of $X/E$ under a definable map $f_j$.
- Applying this result to definably compact $G$ and using strong definable choice on the sets given by the preimages of the $f_j$'s, we have one-dimensional subsets of $G$.
- We prove a general result that any one-dimensional equivalence relation can be eliminated – that is, if $\dim(X/E) = 1$, then $X/E$ is in definable bijection with a one-dimensional definable set.
- Thus, any one-dimensional subset of $G$ is in definable bijection with a one-dimensional subset of $M$.
- We want these one-dimensional subsets to be embedded in definable groups, so we can definably choose representatives of each equivalence class in $G$.
- We then prove that if $f : I \times J \to M$ is a definable function, monotonic in both coordinates, then either $I$ or $J$ can be definably embedded in a definable one-dimensional group.
- Applying to the group operation on $I_j$ yields the desired result.

## What to Expect

In this talk, I will:

- define the topology;
- sketch the proof that one-dimensional quotients can be eliminated;
- give some idea why if $f : I \times J \to M$ is a definable function monotonic in both coordinates, then either $I$ or $J$ can be definably embedded in a definable one-dimensional group.

## Topology

The definition of the topology depends on the following:

**Lemma**

*Let $X$ be a definable set and $E$ a definable equivalence relation on $X$. Then there are definable $Y$ and $E'$ such that $X/E = Y/E'$ and $Y$ admits a partition into finitely many definable sets, $U_1, \ldots, U_m$, respecting $E'$, such that in each set, all equivalence classes have dimension $d$ and projection onto the first $d$ coordinates is a homeomorphism. Moreover, each $U_i$ is an open subset of $M^{k_i}$.*

Thus, from now on, we will assume that after a finite partition, all equivalence classes have the same homeomorphism type, and the base set $X$ is open in its ambient space.

- We suppose that each equivalence class is open in the first $d$ coordinates. Then for each $x \in \pi_{\leq d}(U) \subseteq M^n$, the fiber of $U$ above $x$ has a single representative in each $E$-class.
- For $u \in U$, let $U(u)$ be the fiber of $U$ above $u$.
- Let $u = \langle x', x'' \rangle$ be a generic element of $U$, and let $\mathcal{V}$ be a definable basis of neighborhoods of $x''$, all contained in $U(u)$. Then the family $\mathcal{B} = \{g\mathcal{V} : g \in G\}$ is a basis for a topology ($t$-topology) making $G$ into a topological group.
- The $t$-topology makes $G$ into a topological group because it comes from the usual order topology, so there is a canonical homeomorphism between a neighborhood $V$ of $x''$ in $U(u)$ and a $t$-neighborhood of $u$.
- Thus, definable maps from $G$ to $M^d$ and $M^k$ to $G$ are continuous at generic points, since we may actually consider them to be coming from/going to $U(g)$ for $g$ generic in $U$.
- By methods of Maříková, this shows that $G$ is a topological group with the $t$-topology.

We do not have a finite atlas (yet) on $G$ with this topology. However, what we have is not too bad:

### Proposition

*There are finitely many $t$-open definable sets $W_1, \dots, W_k$ whose union covers $G$. Each $W_i$ is the (non-injective!) image of $\mathcal{U}_0$, where $\mathcal{U}_0$ is a finite disjoint union of definable open subsets of various $M^{r_i}$'s.*

This implies that every definable subset of $G$ has finitely many definably connected components, and thus that many properties of definable groups in o-minimal structures still hold.
In particular, this is enough for the definably simple non-abelian case, with [PPS]'s arguments.

## One-dimensional interpretable sets

- The proof for definably compact groups goes by first showing that definably compact groups have strong definable choice.
- This then allows us to definably pick one-dimensional *interpretable* sets in the group $G$, into whose cartesian product we can suppose that $G$ is embedded.
- Thus, if we can show that these one-dimensional interpretable sets are actually definable and embeddable in one-dimensional groups, we will be done.

### Theorem

*Let $T \subset M^{eq}$ have dimension 1. Then there exists a definable injective map $f : T \to M^m$ for some $m$.*

We consider $\{X_t : t \in T\}$ a definable family, with $T \subset M^{\mathrm{eq}}$ and $\dim T = 1$, and show that the desired map exists for this $T$, by induction on the ambient space of the $X_t$'s. Then we are done by considering $\{[t] : t \in T\}$.

- We perform o-minimal tricks to make all the $X_t$'s cells in $M^k$ of the same dimension $r$.
- If $r = k$, then each $X_t$ is uniquely determined by its "boundary cells," and we are done by induction.
- There are two kinds of points in the $X_t$'s – those that belong to only finitely many $X_t$, and the others. We partition each $X_t$ into these two sets, $X_t^0$ and $X_t'$.
- The union of all $X_t'$ has dimension less than $k$, by straightforward dimension arguments, so it is done by induction.

- Further partitioning $X_t^0$, we can suppose that it is the graph of a function $f_t$ on a cell $C_t$, with distinct $X_t^0$'s disjoint.
- By induction, we have the desired function for the family $\{C_t : t \in T'\}$, where $T'$ is $T$ modulo the equivalence relation $C_s = C_t$. So we need to separate out $X_t$'s projecting to the same $C_t$.
- For each $C_t$, if only finitely many $X_t^0$ project onto $C_t$, then we can take care of them.
- If infinitely many $X_t^0$ project onto $C_t$, then since $\dim T = 1$, there are only finitely many such $C_t$. For each one, we can fix $\bar{a} \in C_t$, and define $g(t) = f_t(\bar{a})$.
- (This step fails in higher dimension, since we would have to pick infinitely many such points.)

## Group-intervals

- We have now reduced the problem of definably compact $G$ to showing that one-dimensional definable subsets of $G$ embed in definable groups.
- Every point of such a set is non-trivial (has a definable group chunk) around it. But we need a group chunk that contains the whole set, up to a finite partition.

### Definition

Let $I$ be a gp-short interval if after a finite partition, it can be definably endowed with the structure of a group chunk, with 0 either an endpoint of $I$ or in $I$.

### Lemma

*Let $\{I_t : t \in T\}$ be a definable family of gp-short intervals, all with the same left endpoint. Then $\bigcup_t I_t$ is a gp-short interval.*

No demands are made on how the group chunks on $I, I_t$ are defined.

**Proof:**

- Let $(a, b) = \bigcup_t I_t$. If we can find $c \in (a, b)$ such that $(c, b)$ is a group interval, then we will be done, since some $I_t$ contains $(a, c)$.
- If there is $c \in (a, b)$ with a definable injection from $(c, b)$ to $(d, e)$ for some $a < d < e < b$, again we are done.
- Thus, we may assume that there are no such maps for any $c$, and thus that our structure has no "poles," treating $b$ like $\infty$.
- This allows us to pick a nonstandard $c < b$, show that $(a, c)$ is gp-short, and then bring down this group operation to the trace of $(a, c)$ on $M$, which is just $(a, b)$.

## Everything Interesting is gp-short

### Theorem

*Let $I, J$ be intervals, and $f : I \times J \to M$ a definable function strictly monotone in both variables. Then at least one of $I$ or $J$ is gp-short.*

Some steps on the way to the proof:

- If $f : I_1 \times \ldots \times I_k \to J$ is definable with $J$ gp-short and all $I_i$ gp-long, then $f$ is constant at every generic point.
- If $f : I_1 \times I_2 \times J \to M$ is definable with $J$ gp-short but $I_1, I_2$ gp-long, then for generic $a \in I_1 \times I_2$, the function $f(a, -)$ is determined (up to finite) by $f(a, d)$ for any generic $d$.
- If $f : I_1 \times I_2 \times I_3 \to M$ is definable with $I_1, I_2, I_3$ gp-long, then we can partition $I_1, I_2, I_3$ so that the functions $f(a, -)$ and $f(b, -)$ on $I_3$ are identical if they ever have the same value.
- So families of functions parameterized by gp-long intervals are one-dimensional, i.e., locally modular.

- The standard machinery of local modularity gives a group operation around $x_0 \in I$ by $x_1 + x_2 = x_3 \iff f_{x_2}^{-1} f_{x_3} = f_{x_0}^{-1} f_{x_1}$.
- This operation is valid whenever the intervals $(x_1, x_0)$ and $(x_2, x_0)$ are gp-short.
- But being gp-short is not a definable property, so the operation "spills over" onto a longer interval, which is necessarily gp-long, contradiction.

# Applying the Theorem

- By an argument, if $h : I_1 \times \ldots \times I_{k+1} \to M^k$ is a definable map injective in each coordinate separately, then at least one of $I_1, \ldots, I_{k+1}$ is gp-short.
- Let $I$ be a one-dimensional set definable in $G$. Let $f_i : I^i \to G$ be defined by $f_i(x_1, \ldots, x_i) = x_1 \cdots x_i$.
- Take $k \geq 1$ maximal such that $f_k$ is injective on $B$, some cartesian product of gp-long intervals in $I^k$.
- We will find a generic $k+1$-tuple $\langle a_1, \ldots, a_{k+1} \rangle \in I^{k+1}$ and a box $B'$ around it such that $f_{k+1}(B')$ is contained in $f_k(B) \cdot a_{k+1}$.
- This is enough, because then we are mapping a $k+1$-dimensional set injectively in each coordinate into a $k$-dimensional set.
- Define the equivalence relation $E'$ on $I^{k+1}$ by $xE'y \iff f_{k+1}(x) = f_{k+1}(y)$.
- Since $f_{k+1}$ is not injective on any gp-long box, this implies that $[\bar{a}]$ is infinite.
- Because $f_k \restriction B$ is injective, the projection of $[\bar{a}]$ on the $k+1$-coordinate is injective, and so the image of $[\bar{a}]$ contains a gp-long interval, $J$.
- We can take $J$ to be definable over parameters independent from $\bar{a}$. Then we can find a gp-long box $B'$ containing $\bar{a}$ such that every $x \in B'$ has $[x]$ projecting in the $k+1$-coordinate onto $J$, and thus in particular containing $a_{k+1}$.