- Our topic this week will be a concept called "T-convexity," defined by van den Dries and Lewenberg in 1995, in a paper called "T-convexity and tame extensions."
- T-convexity is supposed to generalize the idea of a convex subring of a real closed field.
- Let M be a non-archimedean real closed field. Let V be a convex subring.
- The ring V has some nice properties. It is a valuation ring.
- This means that each element of the field of fractions of V is either in V or its inverse is.
- Then the group $\Gamma = M^\times/V^\times$ is the value group, and $V/\mathfrak{m}$ is the residue field, where $\mathfrak{m}$ is the maximal ideal of V.
- val is the map from $M^\times$ to Γ, and res the map from V to $V/\mathfrak{m}$.
- Moreover, the residue field k is still real-closed.
- The structure (M, V) has quantifier elimination in the language $(+, \cdot, 0, 1, <, V)$ (due to Cherlin and Dickmann).

O-minimal real closed fields

O-minimality was first defined by Pillay and Steinhorn. An ordered structure M is o-minimal if any definable subset of M is a finite union of points and intervals.
O-minimality was intended to be analogous to "minimal" in the stable context. There are similarities, but many important differences. For example, "strongly o-minimal" is equivalent to "o-minimal."
A fundamental fact about o-minimality is the Monotonicity Theorem:

Monotonicity Theorem

Let M be any o-minimal structure, and $f : (a, b) \to M$ any M-definable function on the interval (a, b). Then there are $a = a_0 < a_1 < a_2 < \cdots < a_k = b$ such that on each subinterval (a_i, a_{i+1}) the function f is continuous and monotonic.

Our focus here is going to be o-minimal structures expanding fields, or "o-minimal fields." Note that any o-minimal field is real closed.

Convexity in o-minimal real closed fields

van den Dries and Lewenberg looked for a corresponding notion to convexity in the more general area of o-minimal fields.
Given M an o-minimal field with theory T, if V is an arbitrary convex ring, then V is still a valuation ring, but the residue field k will only be a real closed field – it will not have any of the additional structure.
For this reason, they introduced the notion of "T-convexity":

Definition

Let M be an o-minimal field. A convex subring $V \subseteq M$ is *T-convex* if it is closed under all $\emptyset$-definable continuous total unary functions on M.

T-convexity

A convex subring V is *T-convex* if it is closed under all $\emptyset$-definable continuous total unary functions on M.
It turns out that V will also be closed under all n-ary $\emptyset$-definable continuous total functions on M.
If $R \prec M$ with $R \subseteq V$, then V will even be closed under all n-ary R-definable continuous total functions on M.

The residue field and elementary substructures

The operation res takes the T-convex ring V to the real closed field $\mathrm{res}(V)$.
If $R \subset M$ is a substructure of M as a real closed field, with $R \subset V$ as well, then the map $\mathrm{res} : R \to \mathrm{res}(V)$ is a ring homomorphism between fields, and so injective.
The map res will be surjective if and only if $V = R + \mathfrak{m}$.
Such a surjection then induces the structure of a model of T on the residue field $\mathrm{res}(V)$.

When is res surjective on R?

Theorem

If $R \prec M$ with $R \subset V$, then res is surjective on R if and only if R is maximal among elementary substructures of M with $R \subseteq V$.

One direction is easy:

- If res is surjective, then for any $x \in V$, the field R contains some element $x' \in x + \mathfrak{m}$.
- If $S \prec M$ with $R \subsetneq S \subseteq V$, then let $a \in S \setminus R$.
- There is some $a \in R$ with $a - a' \in \mathfrak{m}$, so $a - a' \in S$.
- Thus $1/(a - a') \in S$, but $1/(a - a') \notin V$. $\Rightarrow\Leftarrow$.

Theorem

If $R \prec M$ with $R \subset V$, then res is surjective on R if and only if R is maximal among elementary substructures of M with $R \subseteq V$.

We have done $\Rightarrow$. Time for $\Leftarrow$.

Lemma

If R is maximal among elementary substructures of M with $R \subseteq V$, then R is cofinal in V.

Proof.

- Suppose that R is not cofinal in V. Then there is some $a \in V$ with $a > R$.
- Let $t(a)$ be any element in $R\langle a\rangle$, with t an L_R-term.
- Fix $c \in R$ with t continuous on (c, ∞).
- Since $a > R$, we know that $a > c + 1$. Define f by $f(x) = t(c + 1)$ for $x \leq c + 1$ and $f(x) = t(x)$ for $x > c + 1$.

□

Maximality of R implies surjectivity of res

- If res is not surjective on R, then there is some $a \in \mathrm{res}(V)$ with $\mathrm{res}^{-1}(a) \cap R = \emptyset$.
- We consider $R\langle a\rangle$. Let $t(a)$ be any element in $R\langle a\rangle$, with t an L_R-term.
- We can suppose that t is a continuous function on an interval (c, d) with $c, d \in R \cup \{\pm\infty\}$.
- If $(c, d) = (-\infty, \infty)$, then t is continuous on M, so $t(a) \in V$.
- If $c > -\infty$, then since $\mathrm{res}(c) \neq \mathrm{res}(a)$, we have $1/|a - c| \in V$. Likewise, if $d < \infty$ then $1/|d - a| \in V$.
- Since R is cofinal in V, we can find $\epsilon < 1/|a - c|, 1/|d - a|$ with $\epsilon \in R^+$.
- Define f by $f(x) = t(c + \epsilon)$ for $x \leq c + \epsilon$, $f(x) = t(x)$ for $x \in (c + \epsilon, d - \epsilon)$, and $f(x) = t(d - \epsilon)$ for $x \geq d - \epsilon$.

Maximal elementary substructures of M

- It is not too hard to see that if R_1 and R_2 are two such maximal elementary substructures of M contained in V, then there is an isomorphism between them:
- for any $a \in R_1$, there is a unique $a' \in R_2$ with $a - a' \in \mathfrak{m}$.
- This isomorphism commutes with the res map from each R_i onto $\text{res}(V)$.
- Thus, we have a canonical way to make $\text{res}(V)$ into a model of T, and even to consider it as an elementary substructure of M.

Quantifier elimination for T_{convex}

The theory T_{convex} is just T, the theory of M, together with the unary predicate V and the (infinitely many) statements that V is T-convex.

Theorem

If T is universally axiomatizable and has quantifier elimination, then T_{convex} also has quantifier elimination. T_{convex} is complete. If T is model complete, so is T_{convex}.

The main result here is that T_{convex} has quantifier elimination.

T_{convex} has quantifier elimination

- van den Dries is enamored with a variant of the "Robinson-Shoenfield" test for quantifier elimination, which has two conditions:
  1. Each substructure (R, V) of a model of T_{convex} has a T_{convex}-closure $(\tilde{R}, \tilde{V})$, i.e. $(R, V) \subseteq (\tilde{R}, \tilde{V})$ and $(\tilde{R}, \tilde{V})$ embeds over (R, V) into every model of T_{convex} extending (R, V);
- Satisfying (1) is not hard, since any substructure of a model of T_{convex} is a model of T together with a T-convex subring. If the subring is proper, we are done, and if not, we can adjoin an element larger than R to R, while keeping V fixed, yielding a model of T_{convex}.

T_{convex} has quantifier elimination

- The second condition for the Robinson-Shoenfield test:
  2. If $(R, V) \subseteq (R_1, V_1)$ are models of T_{convex} with $R \neq R_1$, there is an $a \in R_1 \setminus R$ such that $(R\langle a\rangle, V_1 \cap R\langle a\rangle)$ can be embedded over (R, V) into some elementary extension of (R, V).
- This follows essentially from the fact that, given a T-convex subring V of $R \prec S$ and an element $a \in S$ with $|V| < a < |R \setminus V|$, there are exactly two T-convex subrings W of $R\langle a\rangle$ with $(R, V) \subseteq (R\langle a\rangle, W)$ – one that contains a and one that does not.
- This ends the proof. An easy consequence is that T_{convex} is weakly o-minimal.
- Any T_{convex}-definable subset of M is a finite Boolean combination of T-definable sets and sets of the form $\{x : f(x) \in V\}$ for T-definable functions f.

The value group

- We now construct a language L_{vg} that we will use in constructing a theory on Γ, the value group for the valued field M.
- Let Σ be the collection of all L_{convex}-formulas φ with the properties: $T_{\mathrm{convex}} \vdash \forall \vec{y}(\varphi(\vec{y}) \to y_i \neq 0)$ for $i = 1, \ldots, n$, and $T_{\mathrm{convex}} \vdash \forall \vec{x}\vec{y}((\varphi(\vec{x}) \wedge \bigwedge_{i \leq n} \mathrm{val}(x_i) = \mathrm{val}(y_i)) \to \varphi(\vec{y}))$.
- For each $\varphi \in \Sigma$ we add an n-ary predicate U_φ to L_{vg}.
- The interpretation of U_φ is that $\Gamma \models U_\varphi(\gamma_1, \ldots, \gamma_n)$ if and only if there are $a_1, \ldots, a_n \in M$ with $(M, V) \models \varphi(a_1, \ldots, a_n)$ and $\mathrm{val}(a_i) = \gamma_i$ for $i = 1, \ldots, n$.
- Γ_{vg} denotes Γ as an L_{vg}-structure.

T_{vg} is weakly o-minimal

This follows easily from the fact that T_{convex} is weakly o-minimal. But there is a stronger result:

Theorem

If T is power-bounded, T_{vg} is o-minimal. Moreover, up to an extension by definitions, T_{vg} is just the theory of nontrivial ordered vector spaces over K, the "field of exponents of T."

Showing that T_{vg} is o-minimal follows from a very elegant lemma:

Lemma

Let $f : V \to R$ be definable in (M, V). Then $v(f(x))$ is ultimately constant: for some $\gamma \in \Gamma \cup \{\infty\}$ and $a \in V$, we have $v(f(x)) = \gamma$ for all $x \in V$ with $x > a$.

All functions are ultimately constant

Lemma

Let $f : V \to R$ be definable in (M, V). Then $v(f(x))$ is ultimately constant: for some $\gamma \in \Gamma \cup \{\infty\}$ and $a \in V$, we have $v(f(x)) = \gamma$ for all $x \in V$ with $x > a$.

To prove this lemma, we will use a fact:

Fact

Let X and Y be linearly ordered sets such that Y has no largest element. Let $f : X \to Y$ be a non-decreasing function such that $f(X)$ is cofinal in Y. Then $\mathrm{cofinality}(X) = \mathrm{cofinality}(Y)$.

Proof.

Define an equivalence relation E on X by aEb if and only if $f(a) = f(b)$. Let S be a set of representatives for this equivalence relation. Then $\mathrm{cofinality}(X) = \mathrm{cofinality}(S) = \mathrm{cofinality}(f(S)) = \mathrm{cofinality}(Y)$. □

All functions are ultimately constant: a nice model

Lemma

Let $f : V \to R$ be definable in (M, V). Then $v(f(x))$ is ultimately constant: for some $\gamma \in \Gamma \cup \{\infty\}$ and $a \in V$, we have $v(f(x)) = \gamma$ for all $x \in V$ with $x > a$.

- We will want to use a certain well-behaved model of T_{convex}.
- Given M a model of T, we can adjoin an "infinitely large" element t, giving us the model $M\langle t \rangle$, whose elements can be thought of as the germs of M-definable functions near ∞.
- There is a proper T-convex subring $\mathrm{Fin}_M(M\langle t \rangle) = \{f \in M\langle t \rangle : |f| < a \text{ for some } a \in M\}$.
- We will prove the lemma in the structure $(M\langle t \rangle, \mathrm{Fin}_M(M\langle t \rangle))$, which has value group isomorphic to K, the field of exponents of M.

All functions are ultimately constant: proof

Lemma

Let $f : V \to R$ be definable in (M, V). Then $v(f(x))$ is ultimately constant.

Proof.

- We are working in $(M\langle t\rangle, \mathrm{Fin}_M(M\langle t\rangle))$.
- It is not too hard to show that we may take f to be positive and strictly increasing on V, so $v(f(x))$ is decreasing on V.
- If the desired property does not hold, then the set $\Delta = \{v(f(x)) : x \in V\} \subseteq K$ has no smallest element, so by the previous fact, we know that $\mathrm{cofinality}(M) = \mathrm{cofinality}(V) = \mathrm{downward\ cofinality}(\Delta)$.
- But if we replace M by an elementary extension M' with cofinality larger than $|K|$, we have a contradiction.

□

T_{vg} is o-minimal and all functions are piecewise-linear

- Using the lemma, we can show that T_{vg} is o-minimal, since given any definable set in (M, V), we can express it using functions, and then see that these functions must have infema in Γ.
- The fact that T_{vg} is essentially the theory of ordered vector spaces comes from the fact that all definable functions in T_{vg} are piecewise-linear.
- Then a result of Loveys and Peterzil implies the conclusion.
- Piecewise-linearity comes directly from power-boundedness and the exponential-power-bounded dichotomy in o-minimal theories.

Power-boundedness and piecewise-linearity

- Any definable function ϕ in T_{vg} can be "lifted" to a definable function f in T_{convex}.
- This is by definable choice for T_{convex} – for each $x \in V$, we must choose some y with $\mathrm{val}(y) = \phi(\mathrm{val}(x))$.
- By power-boundedness, there is some $a \in M \setminus \{0\}$ and some $\lambda \in K$ such that $\lim_{x\to\infty} f(x)/x^{\lambda} = a$.
- Hence $\phi(\gamma) = \lambda \cdot \gamma + \mathrm{val}(a)$ for all sufficiently small $\gamma \in \Gamma$.
- Mapping finite intervals to intervals near ∞, we can show that above and below every point $\alpha \in \Gamma$ is an interval on which ϕ is K-linear.
- We can then show that the set of points in Γ that do not have an interval around them on which ϕ is K-linear is finite.
- Finally, the intervals around $\alpha \in \Gamma$ can be "glued" together, except at this finite set of bad points.

Applications to preparation theorems

The fact that all definable functions in T_{vg} are piecewise-linear has a very nice consequence, in the form of a preparation theorem:

Theorem

Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be definable in a polynomially-bounded o-minimal structure on $\mathbb{R}$. Then there is a definable finite covering $\mathcal{C}$ of $\mathbb{R}^{n+1}$, and for each $S \in \mathcal{C}$ there are exponents λ and functions $\theta, a : \mathbb{R}^n \to \mathbb{R}$ and $u : \mathbb{R}^{n+1} \to \mathbb{R}$, all definable, such that $\mathrm{graph}(\theta)$ is disjoint from S and $f(x, y) = |y - \theta(x)|^{\lambda} a(x) u(x, y)$, with $|u(x, y)| \le 1/2$.

Tame extensions

- Fix $R \prec M$, maximal in V.
- Any element in $M \setminus R$ is "infinitesimal" over R – either infinitesimally close to an element of R, or close to $\pm\infty$.
- The type over R of any element of $M \setminus R$ is thus definable.
- How about types of tuples of elements of $M \setminus R$?
- The Marker-Steinhorn theorem on definable types gives us the answer:

Theorem
Let $M \prec N$ be an elementary pair of o-minimal structures. If M is Dedekind complete in N, then N realizes only definable n-types over M for all n.

- A consequence of this theorem is the following:

Theorem
Let $M \prec N$ be an elementary pair of o-minimal structures, with M Dedekind complete in N. If $A \subseteq N^n$ is any N-definable set, then $A \cap M^n$ is M-definable.

Stable embeddedness

Theorem
If $A \subseteq M^n$ is definable in (M, V), then $\mathrm{res}(A) \subseteq \mathrm{res}(V)^n$ is definable in the T-model $\mathrm{res}(V)$.

- To prove this, we will first show that $A \cap R^n$ is definable in R for R a maximal elementary substructure of M contained in V.
- By quantifier elimination, A is a Boolean combination of T-definable sets and sets of the form $\{x : f(x) \in V\}$ for T-definable functions f.
- The Marker-Steinhorn theorem already tells us that the intersection of a T-definable set with R^n will be R-definable.
- Thus, it only remains to show that sets of the form $\{x : f(x) \in V\} \cap R^n$ are R-definable.

Definability of types gives tameness

Theorem
Let $M \prec N$ be an elementary pair of o-minimal structures, with M Dedekind complete in N. If $A \subseteq N^n$ is any N-definable set, then $A \cap M^n$ is M-definable.

- This theorem has important consequences for the T_{convex} situation. Namely,

Theorem
If $A \subseteq M^n$ is definable in (M, V), then $\mathrm{res}(S) \subseteq \mathrm{res}(V)^n$ is definable in the T-model $\mathrm{res}(V)$.

- In other words, $\mathrm{res}(V)$ is stably embedded in (M, V), as is R for R any maximal elementary substructure of M contained in V.

Theorem
If $A \subseteq M^n$ is definable in (M, V), then $\mathrm{res}(A) \subseteq \mathrm{res}(V)^n$ is definable in the T-model $\mathrm{res}(V)$.

- We need to show that a set of the form $\{x : f(x) \in V\} \cap R^n$ is R-definable, where f is a T-definable function.
- We have the Marker-Steinhorn theorem:

Theorem
Let $M \prec N$ be an elementary pair of o-minimal structures, with M Dedekind complete in N. If $A \subseteq N^n$ is any N-definable set, then $A \cap M^n$ is M-definable.

- There is an L_R-formula $\phi(x, y)$ such that for any $a \in R^k$ and $b \in R$, we have $R \models \phi(a, b)$ exactly when $|f(a)| < b$.
- Thus, the formula $\exists y \phi(x, y)$ holds in R exactly when $f(x)$ is bounded by some element of R, and thus lies in V.
- Using the isomorphism between R and $\mathrm{res}(V)$, we get our desired definition of $\mathrm{res}(A)$ in $\mathrm{res}(V)$.