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Proof. Since ϕ is stable, $(d_q y)\phi(y; xz)$ is equivalent to a boolean combination of formulas $phi(b_i; xz)$, all of which lie in $\mathrm{dom}(p)$. □

More Products of Types

Lemma 2.4.[7] *If* $p(x)$ *is a* 0*-definable type,* $a \models p$*, and* $q_a(y)$ *is a definable type of the theory* T_a*, then there exists a unique definable type* $r(x, y)$ *such that for any* C*, if* $(a, b) \models r|C$ *then* $a \models p|C$ *and* $b \models q_a|Ca$*.*

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Proof. Given $\phi(xy, z)$, let $\phi^*(x, z)$ be a formula such that $\phi^*(a, z) = (d_{q_a} y)\phi(a, y, z)$. The definition may not be uniform in a, but if ϕ', ϕ'' are two possibilities, then $(d_p x)(\phi' \equiv \phi'')$. Then we can define

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$$(d_r xy)\phi(xy, z) = (d_p x)\phi^*(x, z).$$

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It actually suffices that q_a be definable over $\mathrm{acl}(a)$. This follows by the following lemma.

Almost Done with Products of Types

Lemma 2.5.[7] *Let* $M \subseteq N$ *be models, and let* $\mathrm{tp}(a/N)$ *be* M*-definable. Let* $c \in \mathrm{acl}(Ma)$*. Then* $\mathrm{tp}(ac/N)$ *is definable over* M*. Indeed,* $\mathrm{tp}(a/N) \cup \mathrm{tp}(ac/M) \models \mathrm{tp}(ac/N)$.

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Note that this implies that if q_a is definable over $\mathrm{acl}(a)$, that it is definable over a, since each formula which uses a parameter of $\mathrm{acl}(a)$ is equivalent to one which does not, since the type of every element of $\mathrm{acl}(Ma)$ is definable over Ma.

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Proof. Let $\phi(x,y)$ be a formula over M such that $\phi(a,c)$ holds, and such that $\phi(a,y)$ has m solutions, with m least possible. If $\phi(a,y)$ does not imply a complete type over Na, there exists $\psi(u,x,y)$ over M and $d \in N$ such that $\psi(d,a,y)$ implies $\phi(a,y)$, and $\psi(d,a,y)$ has k solutions with $1 \leq k < m$. Since $\mathrm{tp}(a/N)$ is M-definable, there exists $d' \in M$ satisfying the p-definition of the formulas below, so we have

$$\exists^k(\psi(d',a,y)), \exists^{m-k} y(\phi(a,y) \wedge \neg\psi(d',a,y)).$$

But then either $\psi(d',a,c)$ or $\neg\psi(d',a,c)$, contradicting minimality of m. $\square$

Random, Germs

Notation 2.1. (6) *Given a type p over C with a unique $Aut(\mathbb{U}/C)$-invariant extension $\tilde{p}$ to $\mathbb{U}$, write $a \downarrow_C b$ if $a \models \tilde{p}|\operatorname{acl}(\{b, C\})$.*

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Definition.[4] *Let $A \subseteq M \models T$. Let St_A be the family of all stable, stably embedded A-definable sets – "the stable part of T_A." We write $St_A(c)$ for $A(c) \cap St_A$, where $A(c) = \operatorname{dcl}(A \cup \{c\})$.*

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Definition 2.6.[7,8] *Two definable functions, $f(x, b), g(x, b')$ are said to have* the same p-germ *(for p a definable type) if $\models (d_p x) f(x, b) = g(x, b')$.* The p-germ of $f(x, b)$ is defined over C *if whenever $\operatorname{tp}(b/C) = \operatorname{tp}(b'/C)$, $f(x, b), f(x, b')$ have the same p-germ. Note that the equivalence relation of giving the same p-germ is definable, by considering $f(x, y) = f(x, y')$.*

Stably Dominated Types

Definition 2.8.(8) *A partial type* P *is* stably dominated over C *if there exist* C*-definable maps* $\alpha_i : P \longrightarrow D_i$*,* D *stable,* $\alpha = (\alpha_i)_i$*, such that* $\alpha(a) \downarrow b$ *implies*

$$\mathrm{tp}(b/C\alpha(a)) \models \mathrm{tp}(b/Ca),$$

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We call a definable set D *stable* if every formula $\phi(x; y)$ with $y = (y_1, \ldots, y_m)$ such that $\phi \Rightarrow \bigwedge_{i \leq m} D(y_i)$ is stable. This is often called *stable,stably embedded*. ??

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A type over C is *stably dominated* if it is stably dominated over C via some α.

Proposition 2.9.[8] *Let p be a complete type over $C = \mathrm{acl}(C)$. If p is stably dominated, it has a C-definable extension to $\mathbb{U}$, and this extension is unique.*

Properties of stably dominated types

Most of these are proved in the previous paper, and so are not done here.

Proposition 2.10.[8] *Let* $p = \operatorname{tp}(a/C)$ *be stably dominated.*

1. *(Symmetry) If* $\operatorname{tp}(b/C)$ *is also stably dominated,* $a \downarrow_C b$ *iff* $b \downarrow_C a$*.*
2. *(Transitivity)* $a \downarrow_C bd$ *iff* $a \downarrow_C b$ *and* $a \downarrow_{\operatorname{acl}(Cb)} d$*.*
3. *(Base change) If* $a \downarrow_C b$*, then* $\operatorname{tp}(a/\operatorname{acl}(Cb))$ *is stably dominated.*
4. *If* $\operatorname{tp}(d/C)$ *and* $\operatorname{tp}(b/\operatorname{acl}(Cd))$ *are stably dominated, then so it* $\operatorname{tp}(bd/C)$*. Conversely, if* $a \in \operatorname{dcl}(Cb)$ *and* $\operatorname{tp}(b/C)$ *is stably dominated, so is* $\operatorname{tp}(a/C)$*.*
5. *For any formula* $\phi(x, y)$*,* $(d_p x)(\phi)$ *is a positive boolean combination of formulas* $\phi(a_i, y)$*, where* $a_i \models p|(C \cup \bigcup_{j<i}\{a_j\})$*.*

Metastability

Let Γ be a sort of T which is stably embedded (every subset of Γ^n defined with parameters is definable with parameters in Γ) and orthogonal to the stable part of T (no infinite definable subset of Γ^{eq} is stable).

Definition 1.2.[3] *T is* metastable *(over Γ) if for any partial type P over a base C_0 there exists $C \supset C_0$ and a $*$-definable (over C) map $\gamma_C : P \to \Gamma$ with $\mathrm{tp}(a/\gamma_C(a))$ stably dominated.*

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We will say that C is a *good base for P*. A *good base* is a good base for all partial types over it.

FD

Definition.[4] *We refer to the Morley rank of a stable formula as the (Morley)* dimension*.*
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Definition.[4] *A definable set* X *is* Γ-internal *if* $X \subseteq \mathrm{dcl}(\Gamma, F)$ *for some finite set* F*; equivalently for any* $M \preceq M' \models T$*,* $X(M') \subseteq \mathrm{dcl}(M \cup \Gamma(M'))$*.*

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For the purposes of (FD), it is equivalent to ask that $g(D) \subseteq \Gamma^n$, since Γ eliminates imaginaries.

FD$_\omega$ and Remarks

An addditional hypothesis often used is FD$_\omega$[(4)]: Any set is contained in a good base M which is also a model. Moreover, for any acl-finitely generated $F \subseteq \Gamma$ and $F' \subseteq St_M$ over M, isolated types over $M \cup F \cup F'$ are dense.

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Remarks$^{(4)}$

1. Write $\dim_{st}^{def}(d/B) = \min\{\dim_{st}(D) \mid d \in D, D$ is B-definable$\}$. If $B' = B(d)$, then $\dim_{st}^{def}(B'/B) = \dim_{st}^{def}(d/B)$. Note that we may have $\dim_{st}^{def}(B'/B) > \dim St_B(B'/B)$.

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2. (FD$_\omega$) is true for ACVF, with all imaginary sorts included. (FD) is true for all C-minimal expansions of ACVF.

Strong Germs

Proposition 2.13.[8] *(The strong germ lemma). Let* p *be stably dominated. Assume* p *as well as the* p*-germ of* $f(x, b)$ *are defined over* $C = \mathrm{acl}(C)$*. Then there exists a* C*-definable function* g *with the same* p*-germ as* $f(x, b)$*.*

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Proposition 2.14.[8] *A definable type p is stably dominated iff for any definable function g on p with codomain Γ, the p-germ of g is constant.*

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Proposition 2.14.[8] *A definable type p is stably dominated iff for any definable function g on p with codomain Γ, the p-germ of g is constant.*

Write $g(p)$ for the constant value of the p-germ. The property of p is referred to as *orthogonality of p to Γ*. Note that this is strictly weaker than orthogonality of D to Γ for some definable $D \in p$.

Descent, Question and Answer

Proposition 2.11.[8] *(Descent) Let p, q be $Aut(\mathbb{U}/C)$-invariant $*$-types. Assume that whenever $b \models q|C$, the type $p|Cb$ is stably dominated. Then p is stably dominated.*

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Question 2.12.[8] *Can the descent lemma be proved without the additional hypothesis (E) (that invariant extensions always exist)? Does (E) follow from metastability over an o-minimal Γ?*

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Question 2.12.[(8)] *Can the descent lemma be proved without the additional hypothesis (E) (that invariant extensions always exist)? Does (E) follow from metastability over an o-minimal Γ?*

The answer to the second question is yes.

Proof of Answer

Let p be any type over C_0, some set. We wish to show that p can be extended to an automorphism-invariant type over $\mathbb{U}$. Expand C_0 to a good base for p, C, and let γ_C and α be the maps guaranteed to us by metastability. We have a "decomposition" of p into two types: given $a \models p$, we have $\gamma_C(a)$, and we have $\alpha(a)$. Note that since $\alpha(a)$ is in some stable, stably embedded D, $\mathrm{tp}(\alpha(a))$ has an invariant extension. As well, $\mathrm{tp}(\gamma_C(a)/\alpha(a))$ has an invariant extension: Γ is stably embedded, so we need only extend the type to one over Γ. O-minimal types are categorizable as cuts or noncuts. Each has a simple extreme extension. Thus, $\mathrm{tp}(\alpha(a), \gamma_C(a))$ has an automorphism-invariant extension.

Proof of Answer

Now, consider $q(x, y, z) = \mathrm{tp}((\alpha(a), \gamma_C(a), a)/C)$. Let q' be the extension of q induced by the automorphism-invariant extension of $\mathrm{tp}(\alpha(a), \gamma_C(a))$. I claim that q' is a complete type, and is automorphism-invariant. This is because for any parameter, $\bar{b}$, q implies that $\mathrm{tp}(z/\bar{b}) \subset q'$ is implied by $\mathrm{tp}(xy/\bar{b})$, showing that q' is complete, and, since $\mathrm{tp}(xy/\mathbb{U})$ is automorphism-invariant, so is $\mathrm{tp}(z/\mathbb{U})$, finishing the proof.