Math 1B Midterm 2, July 27 2011, 2:10pm-4:00pm

1. Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{x^n 100^n (n!)^2}{(2n)!}$.

   We use the Ratio Test, so consider

$$\begin{aligned}
&\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}100^{n+1}((n+1)!)^2}{(2n+2)!}}{\frac{x^n 100^n (n!)^2}{(2n)!}} \right| \\
&= \lim_{n\to\infty} \left| \frac{100x(n+1)^2}{(2n+2)(2n+1)} \right| \\
&= 100|x| \lim_{n\to\infty} \left| \frac{n^2+2n+1}{4n^2+6n+2} \right| \\
&= \frac{100|x|}{4}.
\end{aligned}$$

   Thus, $25|x| < 1$ for the Ratio Test to give convergence, and so $|x| < \frac{1}{25}$, and thus the radius of convergence is $\frac{1}{25}$.

2. In Question 1, when $x = -\frac{1}{100}$, how many terms are needed to approximate the sum of the series to within 0.1? Remember that the series starts at $n = 0$.

   When $x = -\frac{1}{100}$, the series is alternating, so we can use the estimation coming from alternating series that the partial sum up to n is at most off by $|a_{n+1}|$. Thus, we want n such that $|a_{n+1}| < 0.1$. This means that $((n+1)!)^2/(2(n+1))! < 0.1$. When $n = 1$, we have $(2!)^2/4! = 4/24 = 1/6$, so this is not sufficient. However, when $n = 2$, we have $(3!)^2/6! = 36/720 = 6/120 = 1/20$, which is sufficient, so $n = 2$, and therefore we need 3 terms.

3. Determine whether the following sequences/series converge (C) or diverge (D). You will lose 3 points for each incorrect answer and gain 3 points for each correct answer, so leave blank if you have no idea. No justification necessary.

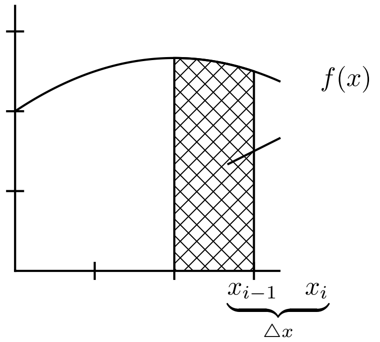
   (a) The sequence $(-1)^n(\ln n)^n/n$;
   (b) The series $\sum_{n=1}^{\infty}(-10)^n/n^{10}$;
   (c) The sequence $(\ln n)^4/n$;
   (d) The series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$;
   (e) The series $\sum_{n=1}^{\infty}(-1)^n \frac{2^n(n^3+1)}{(2^n+1)(n^4-10)}$.

   D (in absolute value, sequence bounded below by $2^n/n$ for sufficiently large n, and this sequence goes to ∞). D (terms do not go to 0, since $10^n > n^{10}$ for $n > 10$). C ($\ln n < n^{1/8}$ for sufficiently large n, so $(\ln n)^4 < \sqrt{n}$, so $(\ln n)^4/n < \frac{1}{\sqrt{n}}$, which goes to 0). C (Comparison Test: $< 2/n^2$ for $n > 2$). C (It is an alternating series, and for sufficiently large n the terms are decreasing towards 0).

4. Find the Taylor series for $\sin x$ centered at the point $a = \pi/2$.

   At $\pi/2$, the derivatives of $\sin x$ are $1, 0, -1, 0, \ldots$. Thus the Taylor series begins $1-(x-\pi/2)^2/2+(x-\pi/2)^4/4!-(x-\pi/2)^6/6!+\ldots$. From this, we can see that the general term is $(-1)^n(x-\pi/2)^{2n}/(2n)!$, and so the Taylor series is $\sum_{n=0}^{\infty}(-1)^n(x-\pi/2)^{2n}/(2n)!$.

5. Let x be a random variable, varying between 0 and 100. Let $f(x)$ be the probability density function of x. We wish to find the formula for the average (mean) value of x. Recall that we found this formula by adding up the values of x that occurred in a large sample of size N and then dividing by N. We can break up the x-axis into $n+1$ equal-size pieces, with endpoints $x_0, x_1, \ldots, x_n$. The number of values of x that occur in a small interval (x_{i-1}, x_i) of length $\triangle x$ is approximately $f(x_i) \cdot N \cdot \triangle x$, and the value of these $f(x_i) \cdot N \cdot \triangle x$ points is approximately x_i.



Number of values in region $\approx f(x_i) \cdot N \cdot \triangle x$

Using this fact, write down a sum that approximates the sum of all the values of x that occur in N. Then write down the actual full formula for the average (mean) value of x. You do not need to derive it from the approximation.

For each piece, the sum of all the values of x that occur in N is $x_i f(x_i) N \triangle x$. Then the sum of all the values of x is $\sum_{i=0}^{n} x_i f(x_i) N \triangle x$. The full formula is $\int_0^{100} x f(x) dx$.

6. Prove that $\ln n \leq 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \leq 1 + \ln n$ for all $n \geq 1$. Hint: how are the functions $\ln x$ and $\frac{1}{x}$ related?

Since $f(x) = 1/x$ is decreasing, positive, and continuous on $(1, \infty)$, we know that $\int_1^n dx/x \leq \sum_{i=1}^{n} \frac{1}{i} = 1 + \sum_{i=2}^{n} \frac{1}{i} \leq 1 + \int_1^n dx/x$. The integral is $\ln n$, and so we have the desired inequalities.

7. Say whether or not $\lim_{x \to 0} \frac{x^4}{\cos x - 1 + x^2/2}$ converges, and find the limit if it does.

Using a Taylor series at 0, we can expand $\cos x$ as $1 - x^2/2 + x^4/4! - x^6/6! + \ldots$. Then the expression becomes $\frac{x^4}{x^4/4! - x^6/6! + \ldots}$, and the limit is $\frac{1}{1/4!} = 24$.