Math 1B Midterm 1, July 8 2011, 3:00pm-4:00pm

1. Evaluate the integral $\int_1^2 x^2 \cdot x^2 \ln(x^2) dx$.

   We have $\int x^4 \ln(x^2)dx = 2\int x^4 \ln(x)dx$. Integrating by parts, let $u = \ln(x)$ and $dv = x^4 dx$. Then $du = dx/x$ and $v = x^5/5$. So the integral is equal to $\ln(x)\cdot x^5/5 - \int \frac{x^4}{5}dx$, which is $\ln(x)\cdot x^5/5 - x^5/25 + C$. Evaluating this indefinite integral at 2 and 1 yields $64\ln(2)/5 - 62/25$.

2. Evaluate the integral $\int_1^{\sqrt{2}} \frac{\sqrt{2-x^2}}{x^2}dx$.

   Let $x = \sqrt{2}\sin(\theta)$, so $dx = \sqrt{2}\cos(\theta)$. Then when $x = 1$, we have $\theta = \pi/4$, and when $x = \sqrt{2}$ we have $\theta = \pi/2$. Our integral then becomes

$$\begin{aligned}
&\int_{\pi/4}^{\pi/2} \frac{\sqrt{2 - 2\sin^2(\theta)}}{2\sin^2(\theta)}\sqrt{2}\cos(\theta)d\theta \\
&= \int_{\pi/4}^{\pi/2} \frac{2\cos^2(\theta)}{2\sin^2(\theta)}d\theta \\
&= \int_{\pi/4}^{\pi/2} \frac{1 - \sin^2(\theta)}{\sin^2(\theta)}d\theta \\
&= \int_{\pi/4}^{\pi/2} \frac{1}{\sin^2(\theta)}d\theta - \int_{\pi/4}^{\pi/2} d\theta \\
&= \int_{\pi/4}^{\pi/2} \csc^2(\theta)d\theta - \theta\Big]_{\pi/4}^{\pi/2} \\
&= -\cot(\theta)\Big]_{\pi/4}^{\pi/2} - \pi/4 \\
&= 1 - \pi/4
\end{aligned}$$

3. Use Simpson's rule with three points to estimate the integral $\int_0^2 \frac{dx}{3^x+1}$. As a function, the fourth derivative of $\frac{1}{3^x+1}$ is always less (in absolute value) than $\frac{1}{3^x+1}$ on the interval $[0, 2]$. Based on this fact and your estimate of the integral, could the actual value of the integral be $\frac{17}{30}$?

   Our function $f(x)$ is $\frac{1}{3^x+1}$. Our three points are $0, 1, 2$, and $f(0) = 1/2$, $f(1) = 1/4$, and $f(2) = 1/10$. Then Simpson's rule gives the approximation to the integral as $\frac{1}{3}\left(\frac{1}{2} + 4\cdot\frac{1}{4} + \frac{1}{10}\right) = \frac{16}{30}$. Since $\frac{1}{3^x+1}$ bounds $f^{(4)}(x)$, we can bound the K term in the error for Simpson's rule by $1/2$. Thus, the error in our estimate is at most

$$\frac{\frac{1}{2}\cdot 2^5}{180\cdot 2^4} = \frac{1}{180}.$$

   Thus the actual value of the integral is within $\frac{1}{180}$ of $\frac{16}{30}$. Since $\frac{17}{30}$ is $\frac{1}{30}$ from $\frac{16}{30}$, the actual value of the integral cannot be $\frac{17}{30}$.

4. Evaluate the integral $\int \frac{4x}{x^2-2x-3}dx$.

   Using partial fractions,

$$\frac{4x}{x^2 - 2x - 3} = \frac{A}{x-3} + \frac{B}{x+1}.$$

Thus $Ax + A + Bx - 3B = 4x$, so $A = 3B$ and $4B = 4$, so $B = 1$ and $A = 3$. Thus our integral is

$$\int \frac{3dx}{x-3} + \int \frac{dx}{x+1} = 3\ln|x-3| + \ln|x+1| + C.$$

5. For what values of p does the integral $\int_1^\infty \frac{dx}{x^p}$ converge? For such values of p, what does the integral evaluate to?

   When $p = 1$, the integral evaluates to $\lim_{t\to\infty}(\ln(x)]_1^t) = \lim_{t\to\infty}(\ln(t)) = \infty$, and so is divergent. When $p \neq 1$, the integral evaluates to

$$\lim_{t\to\infty}(\frac{x^{1-p}}{1-p}]_1^t)$$
$$= \lim_{t\to\infty}(\frac{t^{1-p}}{1-p} + \frac{1}{1-p}).$$

   If $1-p > 0$, then t^{1-p} goes to ∞ as $t \to \infty$, and if $1-p < 0$, then t^{1-p} goes to 0 as $t \to 0$, so $1-p < 0$ for the limit to exist as a finite number. If $1-p < 0$ then $p > 1$, and the limit is equal to $\frac{1}{1-p}$ so the integral is equal to $\frac{1}{1-p}$ when $p > 1$, which is when the integral converges.

6. State the formula for the arc length of a curve given by $x = g(y)$, for $c \leq y \leq d$. Use it to find the arc length of the curve given by $x = \ln(\cos(y))$, for $-\pi/4 \leq y \leq \pi/4$.

   The arc length is given by $\int_c^d \sqrt{1+(g'(y))^2}dy$. When $g(y) = \ln(\cos(y))$, we have $g'(y) = -\tan(y)$. This gives

$$\int_{-\pi/4}^{\pi/4} \sqrt{1+\tan^2(y)}dy$$
$$= \int_{-\pi/4}^{\pi/4} \sec(y)dy$$
$$= \ln(|\sec(y) + \tan(y)|)]_{-\pi/4}^{\pi/4}$$
$$= \ln\left(|\sqrt{2}+1|\right) - \ln\left(|1-\sqrt{2}|\right).$$

7. We want to approximate the area of a surface coming from the revolution of a differentiable curve $y = f(x)$ around the x-axis, as x ranges over an interval $[a, b]$. First, we divide the interval $[a, b]$ into n equal-length subintervals by picking $n+1$ equally-spaced points, $x_0 = a, x_1, \ldots, x_{n-1}, x_n = b$. This gives us $n+1$ points on the curve, $P_0 = (a, f(a)), P_1 = (x_1, f(x_1)), \ldots, P_{n-1} = (x_{n-1}, f(x_{n-1})), P_n = (b, f(b))$. Write down the approximation to surface area that comes from connecting the P_i points by straight line segments, and then revolving these line segments around the x-axis. You should use the fact that if we take a straight line segment of length l, with the left endpoint having y-coordinate r_1 and right endpoint having y-coordinate r_2, then revolving it around the x-axis yields a surface with area $\pi(r_1 + r_2)l$.

   The length of the line segment between P_{i-1} and P_i is $\sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}$. Thus, the surface area of the segment revolved around the x-axis is

$$\pi(f(x_{i-1} + f(x_i))\sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}.$$

   The sum of all these areas is $\sum_{i=1}^n \pi(f(x_{i-1} + f(x_i))\sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}$.