Math 1B Final Exam, August 12 2011, 2:10pm-4:00pm

The full exam is worth 100 points. In each of problems 5-7, you have a choice of two problems to do. However, doing the second problem will only get you 8 points, not the full 16. If you do the first problem, you will end up solving the second problem along the way in any case. In each of problems 5-7, you must indicate which of the problems I should grade, or I will only grade the first one.

Justify all answers and show all work. Clearly mark where your answer is written on your papers.

1. (10 pts) Use Euler's method to approximate the value of $y(1)$ with step size $1/2$ where y satisfies the differential equation $y' = x/y^2$ and $y(0) = 1$.

   $F(x, y) = x/y^2$. We have $x_0 = 0$ and $y_0 = 1$, so $F(0, 1) = 0$. Then $x_1 = 1/2$ and $y_1 = 1$, so $F(1/2, 1) = 1/2$. Then $x_2 = 1$ and $y_1 = 5/4$, so our estimate is $5/4$.

2. (10 pts) A tank contains 1000 L of pure water. Brine that contains 0.05 kg of salt per liter of water enters the tank at a rate of 5 L/min. The tank is thoroughly mixed and drains at the rate of 5 L/min. Fully derive the concentration of salt in the tank after t minutes.

   Let $y(t)$ be the amount of salt in the tank at time t. The rate of change of salt in the tank is given by $5 \cdot 0.05 - 5y/1000$, so $y' = 0.25 - y/200$. We can rewrite this equation as $y' = \frac{1}{4}(1 - y/50)$. It is separable, so we write $\frac{dy}{1-y/50} = \frac{dt}{4}$, in the case that $y \neq 50$. Then $\int \frac{dy}{1-y/50} = \int \frac{dt}{4}$, so $-\ln|1 - y/50| = t/4 + C$, for C an arbitrary constant. Negating and exponentiating both sides, we have $|1 - y/50| = e^{-C}e^{-t/4}$, so $y/50 = 1 - De^{-t/4}$ where D is an arbitrary nonzero constant. Then $y(t) = 50 - De^{-t/4}$. Since $y(0) = 0$, we have $D = 50$, so the amount of salt in the tank after t minutes is $50 - 50e^{-t/4}$ and so the concentration is $\frac{1}{200}(1 - e^{-t/4})$.

3. (10 pts) Is the error in using $1 + 1/4 + 1/9 + \ldots + 1/99^2$ as the approximation to $\sum_{i=1}^{\infty} \frac{1}{n^2}$ less than 10^{-5}? Justify your answer!

   Since the function $f(x) = 1/x^2$ is decreasing, positive, and continuous, and $f(n) = 1/n^2$, we can use the remainder estimate for the integral test, which says that the remainder from using 99 terms is bounded below by $\int_{100}^{\infty} dx/x^2 = \lim_{t\to\infty} -1/t - (-1/100) = 10^{-2}$, which is not less than 10^{-5}.

4. (12 pts) A 2 kg mass is attached to a spring with spring constant 4. The force on the mass exerted by the spring is then $-4x$, where x is the displacement of the mass from its equilibrium position. The system is immersed in a fluid with damping constant 4, so the force on the mass exerted by the fluid is $-4v$, where v is the velocity of the mass. Recall that the velocity v of the mass is the rate of change of the position x of the mass, the acceleration a of the mass is the rate of change of the velocity v, and the total force on the mass is equal to ma. Do both of the following parts:

   (a) Derive the general formula for the motion of the mass as a function of t.

   $F = -4x - 4v$ and $F = 2a$, so $2a = -4x - 4v$ and then $x'' + 2x' + 2x = 0$. This is a homogeneous second-order linear differential equation with constant coefficients, so we solve the auxiliary equation $r^2 + 2r + 2 = 0$ to get $r_1 = -1 + \sqrt{-4}/2$ and $r_2 = -1 - \sqrt{-16}/2$. Since r_1, r_2 are complex, we know that we can write a general solution to the differential equation in the form $e^{-t}(c_1 \cos(t) + c_2 \sin(t))$.

   (b) If at time $t = 0$ the mass is at the equilibrium and at time $t = \pi$ the mass has velocity $4/e^2$, find the position of the mass at any time t.

   Using the result from the first part, $0 = x(0) = c_1$, and $4/e^2 = x'(1) = e^{-1}(c_2 \cos(1) - c_2 \sin(1)) = c_2(e^{-1}(\cos(1) - sin(1))$, so $c_2 = 4/(\cos(1) - \sin(1))$. Thus, the position of the mass at time t is $4e^{-t} \sin(t)/(\cos(1) - \sin(1))$.

5. Find all solutions to the differential equation $y'' - 2y' + y = \frac{e^x x^2}{x^2-1}$.

The solutions to the complementary equation are $y_1 = e^x$ and $y_2 = xe^x$. Thus, the particular solution we are looking for is $y_p = u_1y_1 + u_2y_2$, for some functions u_1, u_2. We want such functions satisfying the two equations $u_1'y_1 + u_2'y_2 = 0$ and $a(u_1'y_1' + u_2'y_2') = G(x)$ with in this case $y_1 = e^x$, $y_2 = xe^x$, $a = 1$, and $G(x) = \frac{e^x x^2}{x^2-1}$. Then we have $u_1'e^x + u_2'xe^x = 0$ and $u_1'e^x + u_2'(x+1)e^x = \frac{e^x x^2}{x^2-1}$. Subtracting the first equation from the second, we get $u_2'e^x = \frac{e^x x^2}{x^2-1}$, and so $u_2' = \frac{x^2}{x^2-1}$. Then $u_2 = \int \frac{x^2 dx}{x^2-1}$, which is $\int dx + \int \frac{dx}{x^2-1}$. The polynomial $x^2 - 1$ factors as $(x-1)(x+1)$, and so we can write $1/(x^2-1) = A/(x-1) + B)/(x+1) = \frac{Ax+Bx+A-B}{x^2-1}$, so $A + B = 0$, $A - B = 1$. This means $2A = 1$ so $A = 1/2$, $B = -1/2$. Then we have $\int \frac{dx}{x^2-1} = \frac{1}{2}(\int dx/(x-1) - \int dx/(x+1)$. The first integral equals $(\ln|x-1|)/2$. The second is $-(\ln|x+1|)/2$, so the full integral is $\frac{1}{2}\ln|\frac{x-1}{x+1}|$, so $u_2 = x + \frac{1}{2}\ln|\frac{x-1}{x+1}|$. Substituting $\frac{x^2}{x^2-1}$ back into the first equation in place of u_2', we have $u_1'e^x + \frac{x^3e^x}{x^2-1} = 0$, so $u_1' = -\frac{x^3}{x^2-1}$. Then $u_1 = -\int \frac{x^3 dx}{x^2-1}$. We can write x^3 as $x^3 - x + x$, so this integral is $-\int \frac{(x^3-x+x)dx}{x^2-1} = -\int xdx - \int \frac{xdx}{x^2-1}$. The second integral is $\ln|x^2-1|/2$, and the first is $x^2/2$. Thus our particular solution is

$$-(x^2 + \ln|x^2-1|)e^x/2 + (x + \frac{1}{2}\ln|\frac{x-1}{x+1}|)xe^x$$

and the general solution is

$$-\ln|x^2-1|e^x/2 + (x + \ln|\frac{x-1}{x+1}|)xe^x/2 + c_1e^x + c_2xe^x.$$

6. (16 pts) Let y be the function satisfying the equation $xy'' - y = 0$, with $y'(0) = 1$. Evaluate $y(-0.1)$ to within 10^{-4}.

Suppose that y can be written as a power series, $\sum_{n=0}^{\infty} c_nx^n$. Then $y'' = \sum_{n=2}^{\infty} n(n-1)c_nx^{n-2}$, so $xy'' - y = \sum_{n=2} n(n-1)c_nx^{n-1} - \sum_{n=0}^{\infty} c_nx^n = \sum_{n=1}(n+1)nc_{n+1}x^n - \sum_{n=0} c_nx^n = 0$, so $c_0 + \sum_{n=1}((n+1)nc_{n+1} - c_n)x^n = 0$, and so $c_0 = 0$ and $(n+1)nc_{n+1} = c_n$, so $c_{n+1} = \frac{c_n}{n(n+1)}$. We write out the first few terms of $y(x)$: $c_1x + c_1x^2/2 + \frac{c_1x^3}{3\cdot 2^2} + \frac{c_1x^4}{4\cdot 3^2\cdot 2^2} + \ldots$. At 0, $y'(0) = c_1 = 1$. Thus, $y(-0.1) = \sum_{n=1}^{\infty} \frac{(-1)^n 10^{-n}}{n((n-1)!)^2}$, which is an alternating series, with terms decreasing toward 0 in absolute value, so if we can find a term less than 10^{-4} in absolute value, we can evaluate up to that term, by the remainder theorem for alternating series. When $n = 3$, we have $10^{-3}/(3\cdot 2^2) < 10^{-4}$, so we can take $-0.1 + 0.1^2/2 = -0.1 + 0.005 = -0.095$.

7. Solve the differential equation $x^2y' + 2xy = \cos^2 x$.

We recognize the left-hand side of this equation as being $(x^2y)'$, so we have $(x^2y)' = \cos^2 x$. Then $x^2y = \int \cos^2 xdx = \int \frac{1+\cos(2x)}{2}dx = x/2 + \sin(2x)/4 + C$, so $y = \frac{1}{2x} + \frac{\sin x\cos x}{2x} + \frac{C}{x^2}$.

8. (10 pts) A parabolic satellite dish is formed by rotating the curve $y = ax^2$ about the y-axis. The dish has a 10-ft diameter, and is 2 ft deep at its center. Find the value of a and the surface area of the dish.

Since the diameter is 10 feet, the maximum value of x is 5, and so since $y = 2$ when $x = 5$, we have $a = 2/25$. Then the surface area of the dish is given by

$$\int_0^5 2\pi x\sqrt{1 + (2ax)^2}dx = \int_0^5 x\sqrt{1 + 4a^2x^2}dx.$$

Letting $u = 1 + 4a^2x^2$, we have $du = 8a^2xdx$, and so the integral becomes

$$\frac{\pi}{4a^2}\int \sqrt{u}du = \frac{\pi}{6a^2}u^{3/2} = \frac{\pi}{6a^2}(1 + 4a^2x^2)^{3/2},$$

which we evaluate at $x = 0$ and $x = 5$, yielding $\frac{625\pi}{6\cdot 4}((1 + 25 \cdot 16/625)^{3/2} - 1) = \frac{625\pi}{24}(\frac{41\sqrt{41}}{125} - \frac{125}{125}) = \frac{5\pi}{24}(41\sqrt{41} - 125)$.